CEGE 4501 Hydrologic Design

Chapter 8:
Hydraulic Processes & Open Channel Flows

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Driven to Discover℠

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Outline

Hydraulic engineering studies liquid flow in pipes and open channels. Pipe flows are typically pressurized flows while open channel flows have free water surfaces and are under atmospheric pressure.

Pipe and open channel flows can be measured in terms of the flow discharge

$$Q = \int_A v \, dA$$

where v is the flow velocity and A represents the cross sectional area. We represent the flow discharge through the above integral equation because in reality the flow profile is not uniformly distributed over the cross section of the channel.

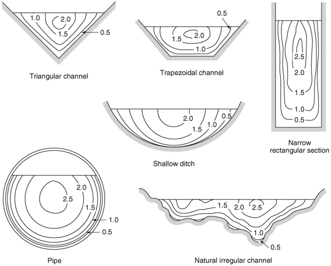


Figure 1: Schematics showing the non-uniform distribution of the flow velocity across the cross section of the flow (from Mays 2011).

Obviously, the commonly used $Q = V\,A$ is an approximation of the flow discharge, where V represents and average velocity. We will use some correction factors when the effects of the non-uniform velocity might be significant.

Conservation of Energy:

There are three sources of energy that move the water parcels

Potential Energy: mgz
Kinetic Energy: $\frac{1}{2}mV^2$
Pressure Energy: $pV = p\frac{m}{\rho}$

$$E = mgz + \frac{1}{2}mV^2 + p\frac{m}{\rho}$$

We often express the above energy per unit weight of the water parcel as follows:

$$\frac{E}{mg} = z + \frac{V^2}{2g} + \frac{p}{\gamma}$$

HGL and EGL

Hydraulic Grade Line (HGL) $= z + \frac{p}{\gamma}$

Energy Grade Line (EGL) $= z + \frac{V^2}{2g} + \frac{p}{\gamma}$

z : elevation head [L]
$\frac{p}{\gamma}$: pressure head [L]
$\frac{V^2}{2g}$: velocity head [L]

There are always some energy dissipation due to viscous stress and friction. These energy losses per unit weight of water are expressed as the head loss h_L.

Therefore the energy equation for two points on the flow path can be written as follows:

$$z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} - h_L = z_2 + \frac{p_2}{\gamma} + \frac{V^2}{2g}$$

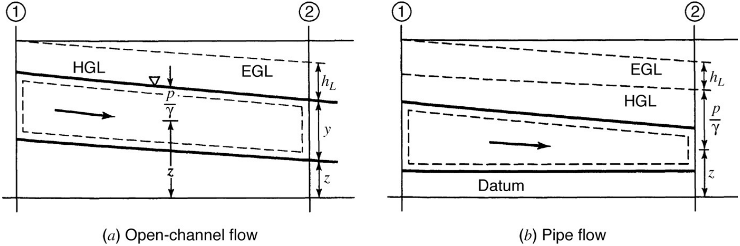


Figure 2: Control volume for open channel (left) and pipe flows (right).

As we discussed and showed in Figure 1, in reality, the velocity profile is not uniform across the flow cross section. To correct for this non-uniformity in the energy equation, we need to account for the variability of the kinetic energy. To that end, we need to recall that the mass of the fluid through a differential area dA per unit time is

$\rho V dA$. Therefore, the total kinetic energy of the flow per unit time is:

$$\frac{1}{2}(\rho v dA)v^2 = \frac{1}{2}\rho v^3 dA$$

Thus, we have

$$\alpha \frac{1}{2}\rho V^3 A = \int_A \frac{1}{2}\rho v^3 dA$$

and

$$\alpha = \frac{1}{AV^3}\int_A v^3 dA$$

where v is the fluid velocity, which varies across the cross section, and V denotes the mean velocity.

Therefore, for a an open channel flow with non-uniform velocity profile we have,

$$z_1 + \frac{p_1}{\gamma} + \alpha \frac{V_1^2}{2g} - h_L = z_2 + \frac{p_2}{\gamma} + \alpha \frac{V^2}{2g}$$

In pipes $\alpha = 2$ for laminar flows and is $\alpha = 1.03 - 1.06$ for turbulent flows. In open channels, α ranges between 1.10 and 1.20 and varies from 1.5 to 2.0 in river flows.

Head loss in pipe flows:

From the Darcy–Weisbach equation, we have,

$$h_L = \frac{f\,L}{D}\frac{V^2}{2g},$$

where h_L is the head loss due to pipe friction, f is the dimensionless friction factor, L is the length of the conduit, D is the diameter, V is the mean flow velocity. For laminar flows $f = \frac{64}{R_e}$, where the Reynold's number $R_e = \frac{VD}{\nu}$. In turbulent flows ($R_e > 2000$), the roughness factor depends on the relative roughness k_s/D, where k_s is the average nonuniform roughness of the pipe. For turbulent flows the friction coefficients are:

smooth pipes:

$$\frac{1}{\sqrt{f}} = 2\log_{10}\left(R_e\sqrt{f}\right) - 0.8$$

rough pipes:

$$\frac{1}{\sqrt{f}} = 2\log_{10}\frac{D}{k_S} + 1.14$$

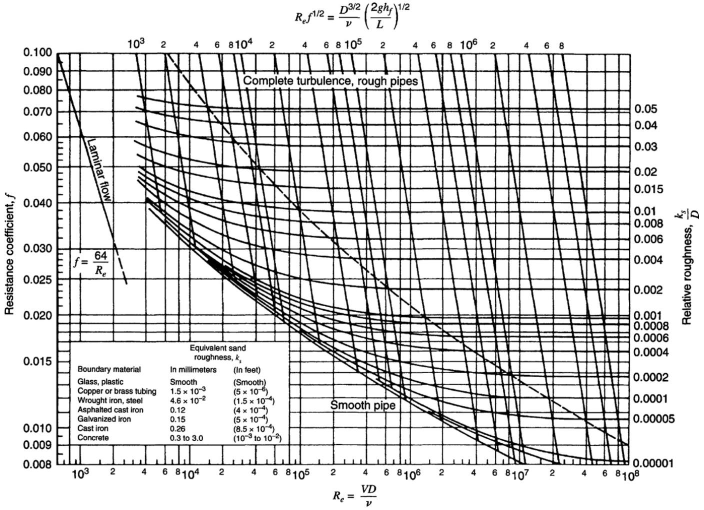


Figure 3: Friction factor (resistance coefficient) f versus R_e from Moody (1944).

Head loss in open channel uniform flows:

In a uniform flow, we have $y_1 = y_2$ and $V_1 = V_2$ and thus

$$h_L = z_2 - z_1 \Rightarrow \frac{h_L}{L} = \frac{z_2 - z_1}{L} = S_f$$

S_f is called friction slope. Because the spatial flow properties does not change in a uniform flow, the friction slope is equal to the slope of the water surface (S_w) and bottom of the channel (S_0).

$$\frac{z_2 - z_1}{L} = S_f = S_0 = S_w$$

Momentum Equation in Open Channel Flow: For a fluid control volume of a uniform or non-uniform flow, we can have the conservation of momentum as follows:

$$\Sigma F = \frac{d(mV)}{dt} = \Sigma(\rho V.A) \times V \quad \text{Newton's second law}$$

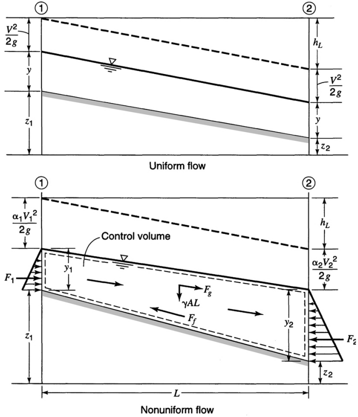


Figure 4: Open channel flow for uniform and non-uniform flow.

when the flow is uniform, we get

$F_g = \gamma AL \sin\theta$

$F_f = \tau_0 PL$

where τ_0 is the shear stress at the bottom of the channel, P is the wetter perimeter, L is the channel length, γ denotes the specific weight, and A is the wetted area of the flow cross section.

As a result, one can obtain $\tau_0 PL = \gamma ALS_0 = \gamma ALS_f$ and thus

$$\tau_0 = \gamma RS_0 = \gamma RS_f$$

where $R = A/P$ is the hydraulic radius.

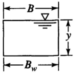
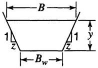
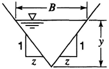
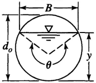
Section:	Rectangle	Trapezoid	Triangle	Circle
				
Area A	$B_w y$	$(B_w + zy)y$	zy^2	$\frac{1}{8}(\theta - \sin\theta)d_o^2$
Wetted perimeter P	$B_w + 2y$	$B_w + 2y\sqrt{1+z^2}$	$2y\sqrt{1+z^2}$	$\frac{1}{2}\theta d_o$
Hydraulic radius R	$\frac{B_w y}{B_w + 2y}$	$\frac{(B_w + zy)y}{B_w + 2y\sqrt{1+z^2}}$	$\frac{zy}{2\sqrt{1+z^2}}$	$\frac{1}{4}\left(1 - \frac{\sin\theta}{\theta}\right)d_o$
Top width B	B_w	$B_w + 2zy$	$2zy$	$\left[\sin\left(\frac{\theta}{2}\right)\right]d_o$ or $2\sqrt{y(d_o - y)}$
$\frac{2dR}{3Rdy} + \frac{1}{A}\frac{dA}{dy}$	$\frac{5B_w + 6y}{3y(B_w + 2y)}$	$\frac{(B_w + 2zy)\left(5B_w + 6y\sqrt{1+z^2}\right) + 4zy^2\sqrt{1+z^2}}{3y(B_w + zy)\left(B_w + 2y\sqrt{1+z^2}\right)}$	$\frac{8}{3y}$	$\frac{4(2\sin\theta + 3\theta - 5\theta\cos\theta)}{3d_o\theta(\theta - \sin\theta)\sin(\theta/2)}$ where $\theta = 2\cos^{-1}\left(1 - \frac{2y}{d_o}\right)$

Figure 5: Geometric function for channel hydraulic properties (Chow 1988).

In fully turbulent flows the shear stress is proportional to the square of the mean flow velocity as follows:

$$\tau_0 = C_f \rho V^2 = \gamma R S_0,$$

where C_f is a dimensionless proportionality constant. As a result, one can obtain,

$$V = \sqrt{\frac{g}{C_f}} \sqrt{RS_0} = C\sqrt{RS_0},$$

where $C = \sqrt{g/C_f}$ is called the Chezy coefficient.

Clearly, in a uniform flow, one can relate the Chezy coefficient to the Darcy-Weisbach roughness coefficient as follows:

$$\frac{h_L}{L} = S_f = S_0 = \frac{f}{4R}\frac{V^2}{2g} \quad \Rightarrow \quad V = \sqrt{\frac{8g}{f}}\sqrt{RS_0} \quad \Rightarrow \quad C = \sqrt{\frac{8g}{f}}$$

Robert Manning (1895) proposed the following experimental formula for the Chezy coefficient:

$$C = \frac{1}{n} R^{1/6}$$

where n is called the Manning roughness coefficient. Therefore, for *a uniform flow* we have

$$V = \frac{1}{n} R^{2/3} S_0^{1/2} \qquad \text{(SI)}$$

$$V = \frac{1.49}{n} R^{2/3} S_0^{1/2} \qquad \text{(EN)}$$

Manning equation is only valid for fully developed turbulent channel flow where $n^6\sqrt{RS_f} \geq 1.1 \times 10^{-13}(SI)$ or $n^6\sqrt{RS_f} \geq 1.9 \times 10^{-13}(EN)$

(*Boldface figures are values generally recommended in design*)

Type of channel and description	Minimum	Normal	Maximum
A. Closed conduits flowing partly full			
A-1. Metal			
a. Brass, smooth	0.009	**0.010**	0.013
b. Steel			
1. Lockbar and welded	0.010	0.012	0.014
2. Riveted and spiral	0.013	0.016	0.017
c. Cast iron			
1. Coated	0.010	0.013	0.014
2. Uncoated	0.011	0.014	0.016
d. Wrought iron			
1. Black	0.012	0.014	0.015
2. Galvanized	0.013	0.016	0.017
e. Corrugated metal			
1. Subdrain	0.017	0.019	0.021
2. Storm drain	0.021	**0.024**	0.030
A -2. Nonmetal			
a. Lucite	0.008	0.009	0.010
b. Glass	0.009	**0.010**	0.013
c. Cement			
1. Neat, surface	0.010	0.011	0.013
2. Mortar	0.011	0.013	0.015

(*Continued*)

Type of channel and description	Minimum	Normal	Maximum
d. Concrete			
1. Culvert, straight and free of debris	0.010	0.011	0.013
2. Culvert with bends, connections, and some debris	0.011	**0.013**	0.014
3. Finished	0.011	0.012	0.014
4. Sewer with manholes, inlet, etc., straight	0.013	0.015	0.017
5. Unfinished, steel form	0.012	0.013	0.014
6. Unfinished, smooth wood form	0.012	**0.014**	0.016
7. Unfinished, rough wood form	0.015	0.017	0.020
e. Wood			
1. Stave	0.010	0.012	0.014
2. Laminated, treated	0.015	0.017	0.020
f. Clay			
1. Common drainage title	0.011	0.013	0.017
2. Vitrified sewer	0.011	0.014	0.017
3. Vitrified sewer with manholes, inlet, etc.	0.013	0.015	0.017
4. Vitrified subdrain with open joint	0.014	0.016	0.018
g. Brickwork			
1. Glazed	0.011	0.013	0.015
2. Lined with cement mortar	0.012	0.015	0.017
h. Sanitary sewers coated with sewage slimes, with bends and connections	0.012	0.013	0.016
i. Paved invert, sewer, smooth bottom	0.016	0.019	0.020
j. Rubble masonry, cemented	0.018	0.025	0.030
B. Lined or built-up channels			
B-1. Metal			
a. Smooth steel surface			
1. Unpainted	0.011	**0.012**	0.014
2. Painted	0.012	0.013	0.017
b. Corrugated	0.021	0.025	0.030
B-2. Nonmetal			
a. Cement			
1. Neat, surface	0.010	0.011	0.013
2. Mortar	0.011	0.013	0.015
b. Wood			
1. Planed, untreated	0.010	0.012	0.014
2. Planed, creosoted	0.011	0.012	0.015
3. Unplaned	0.011	0.013	0.015
4. Plank with battens	0.012	0.015	0.018
5. Lined with roofing paper	0.010	0.014	0.017
c. Concrete			
1. Trowel finish	0.011	**0.013**	0.015
2. Float finish	0.013	0.015	0.016
3. Finished, with gravel on bottom	0.015	0.017	0.020
4. Unfinished	0.014	0.017	0.020
5. Gunite, good section	0.016	0.019	0.023
6. Gunite, wavy section	0.018	0.022	0.025
7. On good excavated rock	0.017	0.020	—
8. On irregular excavated rock	0.022	0.027	—
d. Concrete bottom float finished with sides of			
1. Dressed stone in mortar	0.015	0.017	0.020
2. Random stone in mortar	0.017	0.020	0.024

Figure 6: Manning roughness coefficients. The bold face letters those recommended for design purposes.

Type of channel and description	Minimum	Normal	Maximum
3. Cement rubble masonry, plastered	0.016	0.020	0.024
4. Cement rubble masonry	0.020	0.025	0.030
5. Dry rubble or riprap	0.020	0.030	0.035
e. Gravel bottom with sides of			
1. Formed concrete	0.017	0.020	0.025
2. Random stone in mortar	0.020	0.023	0.026
3. Dry rubble or riprap	0.023	0.033	0.036
f. Brick			
1. Glazed	0.011	**0.013**	0.015
2. In cement mortar	0.012	**0.015**	0.018
g. Masonry			
1. Cemented rubble	0.017	0.025	0.030
2. Dry rubble	0.023	0.032	0.035
h. Dressed ashlar	0.013	0.015	0.017
i. Asphalt			
1. Smooth	0.013	0.013	—
2. Rough	0.016	0.016	—
j. Vegetal lining	0.030	—	0.500
C. Excavated or dredged			
a. Earth, straight and uniform			
1. Clean, recently completed	0.016	0.018	0.020
2. Clean, after weathering	0.018	**0.022**	0.025
3. Gravel, uniform section, clean	0.022	0.025	0.030
4. With short grass, few weeds	0.022	0.027	0.033
b. Earth, winding and sluggish			
1. No vegetation	0.023	0.025	0.030
2. Grass, some weeds	0.025	0.030	0.033
3. Dense weeds or aquatic plants in deep channels	0.030	0.035	0.040
4. Earth bottom and rubble sides	0.028	0.030	0.035
5. Stony bottom and weedy banks	0.025	0.035	0.040
6. Cobble bottom and clean sides	0.030	0.040	0.050
c. Dragline-excavated or dredged			
1. No vegetation	0.025	0.028	0.033
2. Light brush on banks	0.035	0.050	0.060
d. Rock cuts			
1. Smooth and uniform	0.025	0.035	0.040
2. Jagged and irregular	0.035	0.040	0.050
c. Channels not maintained, weeds and brush uncut			
1. Dense weeds, high as flow depth	0.050	0.080	0.120
2. Clean bottom, brush on sides	0.040	0.050	0.080
3. Same, highest stage of flow	0.045	0.070	0.110
4. Dense brush, high stage	0.080	0.100	0.140
D. Natural streams			
D-1. Minor streams (top width at flood stage <100 ft)			
a. Streams on plain			
1. Clean, straight, full stage, no rifts or deep pools	0.025	0.030	0.033
2. Same as above, but more stones and weeds	0.030	0.035	0.040
3. Clean, winding, some pools and shoals	0.033	0.040	0.045
4. Same as above, but some weeds and stones	0.035	0.045	0.050
5. Same as above, lower stages, more ineffective slopes and sections	0.040	0.048	0.055
6. Same as 4, but more stones	0.045	0.050	0.060
7. Sluggish reaches, weedy, deep pools	0.050	0.070	0.080
8. Very weedy reaches, deep pools, or floodways with heavy stand of timber and underbrush	0.075	0.100	0.150
b. Mountain streams, no vegetation in channel, banks usually steep, trees and brush along banks submerged at high stages			
1. Bottom: gravels, cobbles, and few boulders	0.030	0.040	0.050
2. Bottom: cobbles with large boulders	0.040	0.050	0.070
D-2. Flood plains			
a. Pasture, no brush			
1. Short grass	0.025	0.030	0.035
2. High grass	0.030	0.035	0.050
b. Cultivated areas			
1. No crop	0.020	0.030	0.040
2. Mature row crops	0.025	0.035	0.045
3. Mature field crops	0.030	0.040	0.050
c. Brush			
1. Scattered brush, heavy weeds	0.035	0.050	0.070
2. Light brush and trees, in winter	0.035	0.050	0.060
3. Light brush and trees, in summer	0.040	0.060	0.080
4. Medium to dense brush, in winter	0.045	0.070	0.110
5. Medium to dense brush, in summer	0.070	0.100	0.160
d. Trees			
1. Dense willows, summer, straight	0.110	0.150	0.200
2. Cleared land with tree stumps, no sprouts	0.030	0.040	0.050
3. Same as above, but with heavy growth of sprouts	0.050	0.060	0.080
4. Heavy stand of timber, a few down trees, little undergrowth, flood stage below branches	0.080	0.100	0.120
5. Same as above, but with flood stage reaching branches	0.100	0.120	0.100
D-3. Major streams (top width at flood stage > 100 ft). The *n* value is less than that for minor streams of similar description, because banks offer less effective resistance.			
a. Regular section with no boulders or brush	0.025	—	0.060
b. Irregular and rough section	0.035	—	0.100

(Continued)

Source: Chow (1959).

Figure 7: Manning roughness coefficients. The bold face letters are those recommended for design purposes.

Equivalent roughness:

Open channel can be made with different materials with different roughness. The question is: how can we obtain an average roughness coefficient that represents well the flow roughness properties. There are multiple methods. Here, we briefly cover the method by Horton and Einstein.

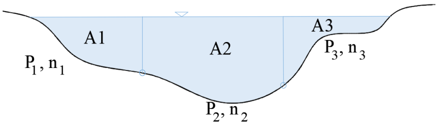


Figure 8: Channel cross section with different roughness properties with P_i and n_i are wetted perimeter and roughness coefficients.

$$V = \frac{1}{n} R^{2/3} S_0^{1/2}$$

$$S_0^{1/2} = \frac{V_1 n_1}{R_1^{2/3}} = \frac{V_2 n_2}{R_2^{2/3}} = \cdots = \frac{V n_e}{R^{2/3}}$$

where V is the average flow velocity and n_e denotes the equivalent roughness. Given that $R_i = \frac{A_i}{P_i}$, we have,

$$\frac{V_i n_i}{R_i^{2/3}} = \frac{V n_e}{R^{2/3}} \Rightarrow \left(\frac{A_i}{A}\right)^{2/3} = \frac{n_i P_i^{2/3}}{n_e P^{2/3}}$$

$$\Sigma A_i = A = A \frac{\Sigma n_i^{3/2} P_i}{n_e^{3/2} P} \Rightarrow n_e = \frac{\left(\Sigma n_i^{3/2} P_i\right)^{2/3}}{P^{2/3}}$$

Best Hydraulic Section:

For a constant area and slope in a channel, the best section, in terms of its maximum ability for flow conveyance, is the one with minimum wetted perimeter because,

$$Q = \frac{1}{n} A R^{2/3} S_0^{1/2} = \frac{1}{n} \frac{A^{5/3}}{P^{2/3}} S_0^{1/2}$$

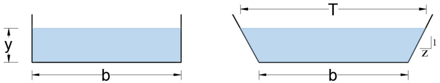


Figure 9: rectangular (left) and trapezoidal(right) cross sections.

For example in a rectangular channel

$P = b + 2y = \frac{A}{y} + 2y$. Setting $\partial P / \partial y = 0$, one can obtain $\frac{-A}{y^2} + 2 = 0$, which results in $y = 2b$.

In a trapezoidal cross section:

$P = b + 2\sqrt{1+z^2}y = \frac{A}{y} - zy + 2y\sqrt{1+z^2}$

$$\begin{cases} \frac{\partial P}{\partial y} = 0 & \Rightarrow b + 2yz = 2y\sqrt{1+z^2} \\ \frac{\partial P}{\partial z} = 0 & \Rightarrow z = \frac{\sqrt{3}}{3} \end{cases}$$

and thus $b = \frac{2y}{\sqrt{3}} = \frac{2\sqrt{3}y}{3}$ and $R = \frac{y}{2}$.

Example: Determine the cross section of the greatest hydraulic efficiency for a trapezoidal channel if the design discharge is 10 m3/s, the channel slope is 0.00052 and $n = 0.025$.

$b = \frac{2\sqrt{3}y}{3} = 1.155y$ and $A = \sqrt{3}y^2 = 1.732y^2$.

$Q = \frac{1}{n} A R^{2/3} S_0^{1/2} = \frac{1}{0.025} (1.732y^2)(\frac{y}{2})^{2/3} 0.00052^{1/2} = 10$

$y = 2.38$ [m] and $b = 2.75$ [m[and $A = 9.81$ [m^2].

Specific Energy:

The total head in open channel flow is:

$$EGL = z + y + \frac{V^2}{2g}$$

in which we assume $\alpha = 1$ and hydrostatic pressure. Using the channel bottom as the datum ($z = 0$) then we define the specific energy as

$$E = y + \frac{V^2}{2g} = y + \frac{Q^2}{2gA^2}$$

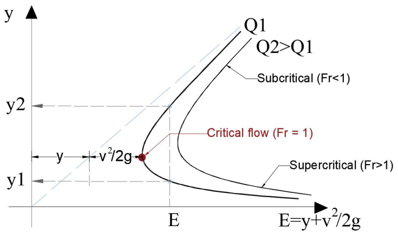


Figure 10: Specific energy versus depth for constant flows. We can see that as $y \to 0$ then $E \to \infty$ and as $y \to \infty$ then $E \to y$.

The minimum point of this curve determines two important flow regimes.

$$\frac{dE}{dy} = 1 - \frac{Q^2}{gA^3}\frac{dA}{dy} = 0$$

where using $T = \frac{dA}{dy}$ as the top width of the flow, one can obtain

$$\frac{Q^2}{g} = \frac{A^3}{T} \quad \Rightarrow \quad \frac{v^2}{g} = \frac{A}{T}$$

Defining a new parameter called the hydraulic depth $D = \frac{A}{T}$, we can conclude that at the minimum specific energy, we have

$$Fr = \frac{v}{\sqrt{gD}} = 1,$$

where Fr is the Froude Number. Therefore, at the minimum point of the specific energy equation $Fr = 1$. From fluid mechanics we know that

$$Fr = \frac{V}{\sqrt{gD}} \begin{cases} < 1 & \text{subcritical flow} \\ = 1 & \text{critical flow} \\ > 1 & \text{supercritical flow} \end{cases}$$

The depth associated with the minimum specific energy is called the *critical depth*. For a rectangular channel isn critical condition, we have $V = \sqrt{gD} = \sqrt{gy_c}$,

$$\frac{q}{y_c} = V = \sqrt{gy_c} \quad \Rightarrow \quad y_c = \left(\frac{q^2}{g}\right)^{1/3}$$

As a result,

$$E_{min} = y_c + \frac{V_c^2}{2g} = y_c + \frac{gy_c}{2g} = \frac{3}{2}y_c.$$

Example: Determine the flow regime and the alternate depth in a rectangular open channel flow when $Q = 20$ [m^3 s^{-1}], b=10 [m], $y = 0.6$ [m].

Solution:

- $V = \frac{Q}{by} = 3.33$ [m^3 s^{-1}]
- $Fr = \frac{V}{\sqrt{gy} = 1.37}$ and thus the flow is supercritical with $y_c = \left(\frac{q^2}{g}\right)^{1/3} = 0.74$ [m].
- $E = y + \frac{Q^2}{2gA^2} = 1.166$ [m]
- To compute the alternate depth we use the equation of energy as $1.166 = y + \frac{2}{gy^2}$ and through a root finding algorithm (e.g., bisection method) one can obtain $y_2 = 0.93$ [m] for the subcritical flow.

Effects of channel bottom humps:

When we have a hump at the bottom of the channel the flow regime might be changed. In this case typically we study two scenarios:

(1) $E_2 = E_1 - \Delta z \geq E_{min}$

In this case explanation of the changes in flow velocity and depth is straightforward. Basically, writing the energy equation for a point right upstream of the hump and a point on the hump, we have $E_1 = E_h + \Delta z$. As a result, the energy over the hump is reduced by Δz and thus when the upstream flow is subcritical, we can see from the curve of the specific energy that the flow depth over the hump decreases and velocity increases. On the other hand, when the flow is supercritical the depth over the hump increases and velocity decreases.

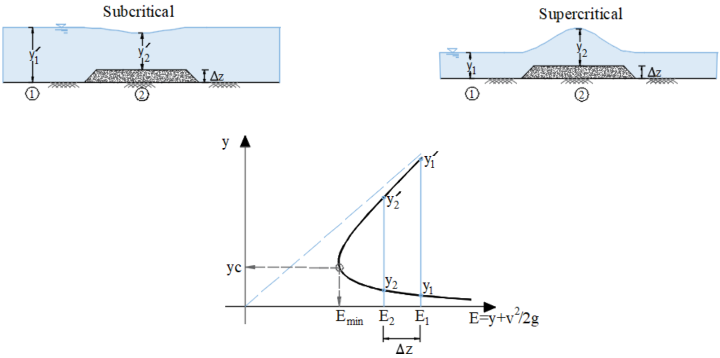


Figure 11: Possible changes in the flow regime when $E_2 = E_1 - \Delta z \geq E_{min}$ for subcritical (left) and supercritical (right) open channel flow.

(2) $E_2 = E_1 - \Delta z \leq E_{min}$. Therefore, for hump heights greater than 0.45, we will have hydraulic jump in the downstream. In this case, the upstream flow needs to adjust its profile to satisfy the conservation of energy. Explanation of the flow profile in subcritical upstream flow is possible. However, when the upstream flow is supercritical the changes in flow profiles can be be explained by the energy equation. In subcritical upstream flow with depth y_1, the flow energy, right upstream of the hump, increases from E_1 to E_2 such that $E_2 = E_{min} + \Delta z$ and thus the flow over the hump will be at critical depth y_c. Right below the hump the flow will have the alternate depth (y_2') associated with the energy of the flow (E_2) with increased upstream depth (y_2) and the flow will be supercritical. Transition of the flow from a supercritical to subcritical regime will occur through a sharp jump in the flow, called hydraulic jump, over which a significant amount of flow energy will be dissipated.

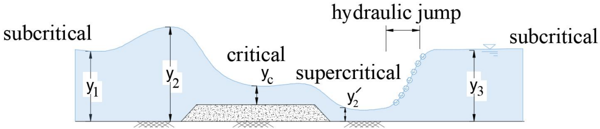


Figure 12: Open channel flow profile due to a bottom hump effect, when $E_2 = E_1 - \Delta z \leq E_{min}$ and the upstream flow is in subcritical regime.

Example: The flow discharge in a channel is $Q = 9.91$ [m^3 s^{-1}] with upstream flow depth of $y_1 = 1.83$ [m] and width $b = 3.05$ [m].

(a) What is the minimum height of a hump to make the flow supercritical over the hump.

(b) Compute y_2 when $\Delta z = \frac{1}{2}\Delta z_c$.

(c) Determine the flow horizontal profile for $\Delta z = 2\Delta z_c$

Solution:

(a) $q = \frac{Q}{b} = 3.25$ [m^2 s^{-1}] and $y_c = \left(\frac{q^2}{g}\right)^{1/3} = 1.025$ [m].

The upstream flow is subcritical as $y_1 > y_c$.

$E_{min} = \frac{3}{2} y_c = 1.54$ [m] and $E_1 = y + \frac{q^2}{2gy^2} = 1.99$ [m].

and thus $\Delta z = E_1 - E_{min} = 0.45$ [m].

(b) $\Delta z = 0.225$ [m]

The specific energy over the hump is $E_h = E_1 - \Delta z = y_h + \frac{q_2^2}{2gy_h^2} = 1.765$, which results in $y_h = 1.53$ [m].

The surface water depression over the hump is

$y_1 - y_h - \Delta z = 1.83 - 1.53 - 0.225 = 0.075$ [m].

(c) $\Delta z = 0.9$ [m]

In this situation since $E_1 - \Delta z < E_{min}$and the upstream flow is in subcritical condition, we have hydraulic jump in the downstream and the flow becomes critical over the hump. Therefore, $y_h = y_{cr}$ abd $E_2 = \Delta z + E_{min}$.

$E_2 = 1.54 + 0.9 = 2.44$ [m] and knowing that $y_c = 1.025$ [m], one can obtain

$y_2 + \frac{q^2}{2gy_2^2} = 2.44 \rightarrow y_2 = 2.35$ [m] and

$y_2' + \frac{q^2}{2gy_2'^2} = 2.44 \rightarrow y_2' = 0.53$ [m].

Conservation of Momentum in Open Channel Flows:

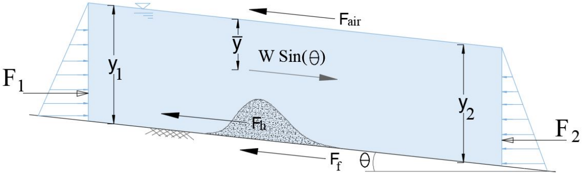


Figure 13: Schematic showing conservation of momentum in an open channel flow.

$$\Sigma F_x = \Sigma_i \beta_i \rho (V_i . A) V_i$$
$$\rho Q(\beta_2 V_2 - \beta_1 V_1)$$

where $\beta = \frac{1}{AV^2} \int_A v^2 dA$ is the momentum correction factor that accounts for nonuniform velocity distribution the flow cross section.

Based on the schematic in Figure 13, we have

$$\Sigma F = F_1 - F_2 - F_f - F_h - F_{air} + W \sin\theta$$

which can be rearranged as follows:

$$F_1 - F_2 - F_{ext} = \rho Q(\beta_2 V_2 - \beta_1 V1),$$

where $F_{ext} = F_f + F_h + F_{air}$,

and $F_1 = \gamma A_1 \overline{y}_1$ and $F_1 = \gamma A_1 \overline{y}_1$.

A: flow cross section

$\overline{y}$: distance of the centroid of the section form the surface

γ: specific weight of the fluid

Therefore,

$$F_{ext} = (\rho Q \beta_2 V_2 + \gamma \overline{y}_2 A_2) - (\rho Q \beta_1 V_1 + \gamma \overline{y}_1 A_1)$$

Assuming $\beta_1 = \beta_2 = 1$

$$\frac{F_{ext}}{\gamma} = F = \left(\frac{Q^2}{gA_2} + \overline{y}_2 A_2 \right) - \left(\frac{Q^2}{gA_1} + \overline{y}_1 A_1 \right)$$

in practice the value of $F_e xt$ could be much smaller than the right hand side of the above equation in this case:

$$\frac{Q^2}{gA_2} + \overline{y}_2 A_2 = \frac{Q^2}{gA_1} + \overline{y}_1 A_1$$

The following quantity is called the *specific force*,

$$F = \frac{Q^2}{gA} + \overline{y} A.$$

Setting $\frac{\partial F}{\partial y} = 0$, one can obtain

$$\frac{\partial F}{\partial y} = -\frac{Q^2}{g} \left(\frac{\frac{dA}{dy}}{A^2} + \frac{d(\overline{y} A)}{dy} \right) = 0$$

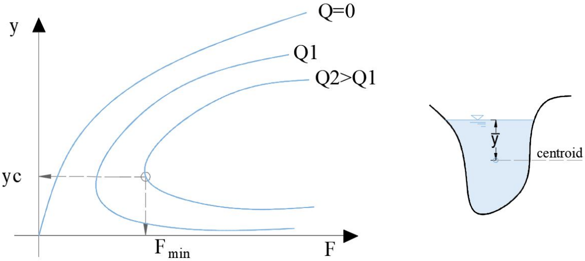


Figure 14: Schematic showing the specific force for different values of flow discharge and the critical flow depth ($Fr = 1$), which is obtained at minimum specific force.

$$\frac{\partial F}{\partial y} = -\frac{Q^2}{g}\left(\frac{T}{A^2} + A\right) = 0,$$

which results in the critical flow condition as,

$$\frac{Q^2}{g} = \frac{A^3}{T}.$$

Therefore, in a uniform flow with a constant discharge, the specific force is minimum at critical flow condition with Froude number 1. We already discussed that when a supercritical flow meets a subcritical flow due to continuity of mass and momentum the flow needs to adjust its shape and a turbulent jump occurs during the transition from super to subcritical flow condition. Since hydraulic jump is often a local phenomenon that occurs over a short reach of the river flow, we can assume that there is no significant external forces acting on the flow in the middle of the jump. In this case, we can assume the specific force before and after the jump remains constant.

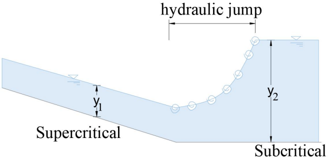


Figure 15: Schematic showing the specific force for different values of flow discharge and the critical flow depth ($Fr = 1$), which is obtained at minimum specific force.

As the specific force remains constant before and after the jump, in a uniform rectangular channel flow the above conjugate depths can be obtained as follows:

$$y_2 = \frac{-y_1}{2} + \sqrt{\left(\frac{y_1}{2}\right)^2 + \frac{2q^2}{gy_1}}$$

$$y_1 = \frac{-y_2}{2} + \sqrt{\left(\frac{y_2}{2}\right)^2 + \frac{2q^2}{gy_2}},$$

where q is the discharge flow per unit width of the flow.

During the hydraulic jump, a significant amount of the flow energy can be lost due to dissipative effects of turbulent eddies. This energy loss can be obtained from the specific energy equation $\Delta E = E_2 - E_1$ as follows:

$$\Delta E = \frac{(y_2 - y_1)^3}{4y_1y_2}.$$