CEGE 4501 Hydrologic Design

Chapter 7:
Unit Hydrograph and Routing

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Driven to Discover℠

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Outline

Linear Reservoir and Convolution I

A linear reservoir is a simple and meaningful representation of a watershed. Continuity equation for a linear reservoir is written as follows:

$$\begin{cases} \frac{dS(t)}{dt} &= I(t) - Q(t) \quad \textit{continuity equation} \\ S(t) &= K\,Q(t) \quad \textit{linear storage function} \end{cases}$$

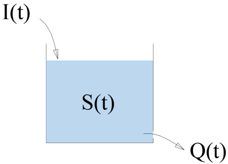

Figure 1: Schematic of a linear reservoir. A reservoir is linear if we assumed that the storage function $S(t)$ is linearly related to the outflow through the constant K.

where $Q(t)$ [L^3 T^{-1}] is the outlfow, $I(t)$ [L^3 T^{-1}] is the inflow, $dS(t)/dt$ [L^3 T^{-1}] is change in storage over time, and K represents a time scale that captures the residence time of the reservior [T]. Now upon substitution, we obtain and solve a *first-order ordinary differential equation (ODE)*:

$$K\frac{dQ(t)}{dt} = I(t) - Q(t)$$

$$\frac{dQ(t)}{dt} = I(t)/K - Q(t)/K$$

Multiplying both sides by $e^{t/K}$, one can get:

$$e^{t/K}\frac{dQ(t)}{dt} + e^{t/K}Q(t)/K = e^{t/k}I(t)/K$$

Now because of the product rule we can rewrite the above equation as follows:

Linear Reservoir and Convolution II

$\frac{d\left(Q(t)e^{t/K}\right)}{dt} = I(t)\frac{1}{K}e^{t/K},$

and thus, we have,

$\int_0^t d\left(Q(s)e^{s/K}\right) = \int_0^t I(s)\,\frac{1}{K}\,e^{s/K}\,ds$

Outflow of a linear reservoir:

$$Q(t) = Q_0 e^{-t/K} + \int_0^t I(s)\frac{1}{K}e^{-(t-s)/K}ds$$

This solution to the differential equation states that the outflow at any time relies on an initial flow (first RHS term) that goes to zero over time and the integral term (second RHS term) that represents the effects of the reservoir resident time on the outflow. The integral term is a *convolution integral*. Learning the concept of convolution is crucial for understanding the concept of the *unit hydrograph*.

Mathematical convolution for two generic functions $f(s)$ and $g(t)$ is defined as follows:

$$f * g(t) = \int_{-\infty}^{+\infty} f(s)\,g(t-s)\,ds = \int_{-\infty}^{+\infty} f(t-s)\,g(s)\,ds$$

Figure 2 contains two toy examples of convolutions to visualize the process. We can see that the convolution of a function $f(t)$ with a kernel $g(s)$ may result in translation (shift) and dilution/diffusion of $f(t)$.

Linear Reservoir and Convolution III

Figure 2: Toy examples of convolution (Credit: Wikipedia).

Now looking back on the integral in the linear reservoir outflow equation, we can see that it is a convolution operator as $I(s)$ resembles $f(s)$ and $\frac{1}{k}e^{-(t-s)/K}$ resembles $g(t-s)$ as $I * u(t) = \int_0^{\infty} I(s)u(t-s)ds$, where $u(t) = \frac{1}{K}\, u^{-t/K}$.

If we define the input as the *unit impulse* function or the Kronecker delta function as follows:

$$I(s) = \begin{cases} 0 & \text{if } s \neq t \\ 1 & \text{if } s = t \end{cases}$$

Then, we can define the impulse response function of a linear reservoir as follows:

$$I * u(t) = u(t-s) = \frac{1}{K}e^{-(t-s)/K}.$$

Similar to the above formalism for a linear reservoir, a watershed has a response function to a unit pulse of excess precipitation—albeit a more complex one. This response function is known as the *unit hydrograph*. More specifically, the unit hydrograph $u(t)$ of a watershed is defined as the *direct runoff hydrograph (DRH)* resulting from a *unit pulse of excess rainfall*, which is defined as a unit depth of excess rainfall (i.e. 1 in., 1 cm, etc) distributed uniformly over the entire watershed area for an effective duration of $\triangle t$.

Linear Reservoir and Convolution IV

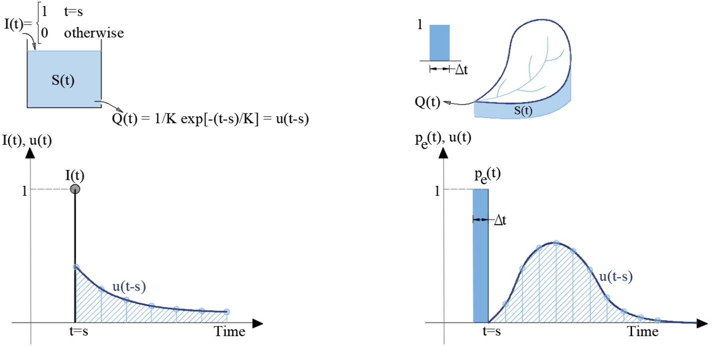

Figure 3: The response of a linear reservoir to a *unit impulse* (left) resembles the response of a watershed to a *unit pulse* of rainfall (right), which is a unit amount of rainfall uniformly distributed over the watershed for an specified duration of $\triangle t$

Convolution is a linear operator

$$f * (a\, g_1(t) + b\, g_2(xt)) = a\, f * g_1(t) + b\, f * g_2(t),$$

which means that we can superimpose the impacts of two pulses of rainfall over a watershed though the linear combination of their individual responses.

Linear Reservoir and Convolution V

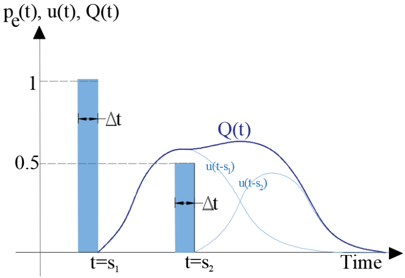

Figure 4: Since we aim to model the watershed response to pulses of rainfall thorough convolution, we can linearly combine the response functions due to linearity of the convolution operator. For example, the above schematic shows that $Q(t) = 1 \times u(t - s_1) + 0.5 \times u(t - s_2)$.

Therefore, having the unit hydrograph, one can obtain the outflow of a watershed as convolution of excess rainfall $p_e(t)$ with the unit hydrograph $u(t)$ as follows:

$$Q(t) = \int_0^t p_e(s)u(t - s)ds.$$

Unit Hydrograph I

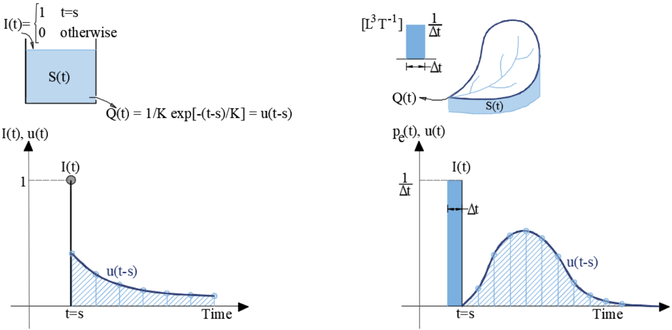

Figure 5: Convolution of Excess Rainfall Hyetograph (ERH) with $m = 1 \dots m$ intervals with the unit hydrograph of a basin, which is characterized at $l = 1 \dots L$ intervals, results in a Direct Runoff hydrograph (DRH) at $n = 1 \dots N$ points.

Since, hydrologic data is not continuous, we must use discretized versions of the convolution equation we discussed previously as follows:

$$Q_n = \sum_{m=1}^{min(n,M)} p_m u_{n-m+1} \qquad n = 1, \dots, N \text{ and } N = M + L - 1.$$

Unit Hydrograph II

For the example shown in Figure 5, we have

$$Q_n = p_1 \, u_n + p_2 \, u_{n-1} + p_3 \, u_{n-2},$$

where $M = 3$, $L = 6$, $N = M + L - 1 = 8$. Therefor, the above convolution can be expressed as a linear system of equation as follows:

$$\begin{aligned}
Q_1 &= p_1 u_1 + 0 + 0 \\
Q_2 &= p_1 u_2 + p_2 u_1 + 0 \\
Q_3 &= p_1 u_3 + p_2 u_2 + p_3 u_3 \\
Q_4 &= p_1 u_4 + p_2 u_3 + p_3 u_2 \\
Q_5 &= p_1 u_5 + p_2 u_4 + p_3 u_3 \\
Q_6 &= p_1 u_6 + p_2 u_5 + p_3 u_4 \\
Q_7 &= 0 + p_2 u_6 + p_3 u_5 \\
Q_8 &= 0 + 0 + p_3 u_6
\end{aligned}$$

Thus one can arrange all the rainfall values in a matrix form as follows:

Unit Hydrograph III

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 & 0 & 0 & 0 \\ p_2 & p_1 & 0 & 0 & 0 & 0 \\ p_3 & p_2 & p_1 & 0 & 0 & 0 \\ 0 & p_3 & p_2 & p_1 & 0 & 0 \\ 0 & 0 & p_3 & p_2 & p_1 & 0 \\ 0 & 0 & 0 & p_3 & p_2 & p_1 \\ 0 & 0 & 0 & 0 & p_3 & p_2 \\ 0 & 0 & 0 & 0 & 0 & p_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

and solve it for u_i values to obtain the unit hydrograph from the observed flow of a precipitation excess event. We can see that in the above linear system of equations we have more equations that unknown. Therefore, calculation of the unit hydrograph is often an overdetermined problem.

Unit Hydrograph IV

As previously explained, the unit hydrograph (UH) is the response function of a linear hydrologic system to a unit pulse of rainfall, which can be convolved with the excess rainfall to simulate the direct runoff. Below is an example of a 6-hour unit hydrograph being convolved with hourly rainfall averaged into 6-hour bins. The example shows the principle of hydrograph superposition.,

It is important to note that by construction, the base time of a DRH resulting from a certain duration ERH is constant, therefore UHs vary with storm duration and for a given watershed, the UH is an unchanged characteristics of a watershed for a specific duration of excess rainfall.

Figure 6: Example of 6-hr Unit Hydrograph convolution (Credit: The COMET Program).

Unit Hydrograph Derivation I

In a general form the convolution of excess rainfall and the unit hydrograph can be expressed as follows:

$$Q_n = \sum_{m=1}^{min(n,M)} p_m u_{n-m+1}$$

Q_n : Direct runoff [L^3 T^{-1}] , $n = 1, ..., N$

p_m : Excess rainfall [L], $m = 1, ..., M$

u_{n-m+1}: Unit hydrograph [L^2 T^{-1}], $l = n - m + 1$.

$$\begin{array}{lllllllllllll}
Q_1 & = & p_1 u_1 & & & & & & & & & & \\
Q_2 & = & p_2 u_1 & + & p_1 u_2 & & & & & & & & \\
Q_3 & = & p_3 u_1 & + & p_2 u_2 & + & p_1 u_3 & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & & & & & \\
Q_M & = & p_M u_1 & + & p_{M-1} u_2 & + & \cdots & + & p_1 u_M & & & & \\
Q_{M+1} & = & 0 & + & p_M u_2 & + & \cdots & + & p_2 u_M & + & p_1 u_{M+1} & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
Q_{N-1} & = & 0 & + & 0 & + & \cdots & + & 0 & + & 0 & +p_M u_{N-M} & + & p_{M-1} u_{N-M+1} \\
Q_N & = & 0 & + & 0 & + & \cdots & + & 0 & + & 0 & +0 & + & p_M u_{N-M+1}
\end{array}$$

Matrix Derivation

The above system of linear equation which is results from a discrete convolution operation can be rewritten in matrix form as follows:

$$\mathbf{q} = [Q_1, Q_2, \ldots, Q_N]^T, \quad \mathbf{u} = [u_1, u_2, \ldots, u_L]^T, \quad L = N - M + 1$$

Unit Hydrograph Derivation II

$$\mathbf{P} = \begin{bmatrix} p_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ p_2 & p_1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ p_3 & p_2 & p_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_M & p_{M-1} & p_{M-2} & p_{M-3} & \dots & p_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & p_M & p_{M-1} & p_{M-2} & \dots & p_2 & p_1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_M & p_{M-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_M \end{bmatrix}$$

where $\mathbf{q}$ is an N-by-1 column vector, $\mathbf{P}$ is an N-by-L matrix, and $\mathbf{u}$ is an L-by-1 column vector.

For example, the above matrix form in Figure 5 can be expressed as follows:

$$\mathbf{Pu} = \mathbf{q}$$

$$\begin{bmatrix} p_1 & 0 & 0 & 0 & 0 & 0 \\ p_2 & p_1 & 0 & 0 & 0 & 0 \\ p_3 & p_2 & p_1 & 0 & 0 & 0 \\ 0 & p_3 & p_2 & p_1 & 0 & 0 \\ 0 & 0 & p_3 & p_2 & p_1 & 0 \\ 0 & 0 & 0 & p_3 & p_2 & p_1 \\ 0 & 0 & 0 & 0 & p_3 & p_2 \\ 0 & 0 & 0 & 0 & 0 & p_3 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{bmatrix}$$

where $M = 3$ is an $L = 6$ and $N = M + L - 1 = 8$.

Unit Hydrograph Derivation III

The matrix form can be easily implemented into software library for numerical linear algebra. As previously explained, the equation $\mathbf{Pu} = \mathbf{q}$ is an over-determined linear system, since we have more equations than unknowns. Two of the main methods used to solve this system of equations for $\mathbf{u}$ are:

1. Forward Substitution: Write out the first L equations, which are in upper triangular form, solve the first equation and sequentially substitute and solve the remaining equations. In this formalism only the first L equations are used and this questions remains unanswered that what happens if we involve other existing equations in our calculation. In other words, because equations are more than unknowns there are multiple solutions to this problem.
2. Least Squares: Using linear algebra, we can derive a formula to calculate the least squares estimate of the unit hydrograph ordinates as follows:

$$\mathbf{Pu} = \mathbf{q}$$

$$\mathbf{P}^T\mathbf{Pu} = \mathbf{P}^T\mathbf{q}$$

$$\mathbf{u} = \left(\mathbf{P}^T\mathbf{P}\right)^{-1}\mathbf{P}^T\mathbf{q}$$

where $(.)^T$ is the matrix transposition operator, which flips a matrix over its diagonal. Note that by least squares, we mean that the solution has minimum error variance amount all other possible solutions.

The above matrix formulation of the linear system can be solved efficiently using a technical programing language such as MATLAB. Note that, in the above formulation, we multiply both sides with $\mathbf{P}^T$ to obtain a square matrix $\mathbf{PP}^T$, which might be invertible. We need to note that $\mathbf{P}$ is often a rectangular matrix with more rows than columns ($N \gg L$) and can not be inverted. The matrix $\left(\mathbf{P}^T\mathbf{P}\right)^{-1}\mathbf{P}^T$ is called pseudo inverse of $\mathbf{P}$. In MATLAB, computation of the unit hydrograph can be simply implelmented using eaither of the following commands: `u = P\q` or `u = pinv(P)*q`. The first command is preferred, though.

Unit Hydrograph Derivation IV

S-Hydrograph Method

As we mentioned the UH is a time invariant characteristic of a watershed in response to unit pulse of excess rainfall with an specific duration. However, since the UH is obtained trough a linear process, we can change a unit hydrograph form one duration to another. To that end, we need to derive the *S-hydrograph* based on the principle of superposition.

By definition, the S-hydrograph is the direct runoff hydrograph of a watershed to a continuous train of pulses of rainfall with duration Δt.

In other words, the S-hydrograph $g(t) = [u(t) + u(t - \Delta t) + u(t - 2\Delta t) + \ldots]$ is the summation of an infinite number of unit pulse of excess rainfall with duration Δt as illustrated in Figure 7.

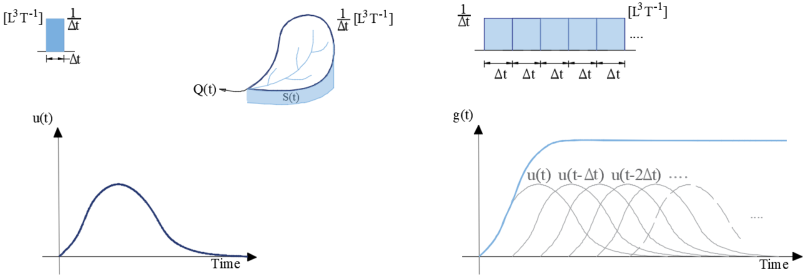

Figure 7: Conceptual representation of the S-hydrograph.

Unit Hydrograph Derivation V

As a result, the unit hydrograph for $\Delta t'$ duration, which must be an integer multiplier of Δt, can be obtained as follows:

$$u\left(\Delta t'\right) = \frac{1}{\Delta t'}\left[g(t) - g(t - \Delta t')\right]$$

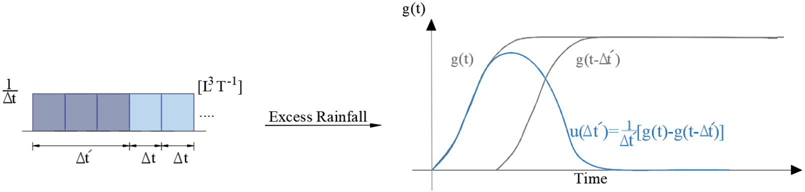

Figure 8: Derivation of the unit hydrograph with duration $\Delta t' = n\Delta t$, where $n \in \mathbb{N}$, from an S-hydrograph that is generated from summation of unit hydrographs with duration Δt.

Unit Hydrograph Derivation VI

Synthetic Unit Hydrographs

Thus far, the methods we have described only apply for the basin with gage measurements of the streamflow, however, many basins are ungauged. Therefore, engineers have analyzed many unit hydrographs from gauged basins to obtain a universal functional representation for the unit hydrograph that can be generalized to ungauges basins. This representation of the unit hydrograph is often called *synthetic unit hydrograph (SUH)* that can be applied to ungauged basins. These synthetic methods typically scale a general UH shape or calculate key ordinates based on easily determined watershed and excess precipitation characteristics. There are many of these methods, such as Snyder's or Clark's SUH, however, we will cover the very popular *NRCS (SCS) dimensionless synthetic hydrograph* method. The dimensionless SCS hydrograph and its triangular representation are shown in Figure 7 and tabulated in Table 1.

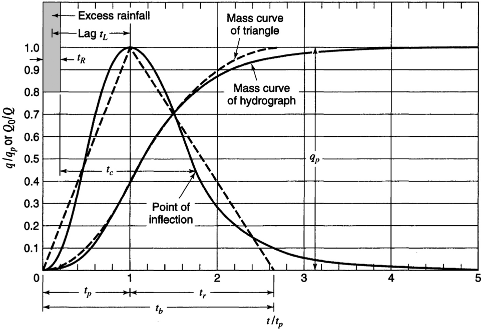

Figure 9: Dimensionless curvilinear and equivalent triangular SCS unit hydrographs (USDA, 1986)

Unit Hydrograph Derivation VII

Time ratios t/t_p	Discharge ratios q/q_p	Mass curve ratios Q_a/Q
0	0.000	0.000
0.1	0.030	0.001
0.2	0.100	0.006
0.3	0.190	0.012
0.4	0.310	0.035
0.5	0.470	0.065
0.6	0.660	0.107
0.7	0.820	0.163
0.8	0.930	0.228
0.9	0.990	0.300
1.0	1.000	0.375
1.1	0.990	0.450
1.2	0.930	0.522
1.3	0.860	0.589
1.4	0.780	0.650
1.5	0.680	0.700
1.6	0.560	0.751
1.7	0.460	0.790
1.8	0.390	0.822
1.9	0.330	0.849
2.0	0.280	0.871
2.2	0.207	0.908
2.4	0.147	0.934
2.6	0.107	0.953
2.8	0.077	0.967
3.0	0.055	0.977
3.2	0.040	0.984
3.4	0.029	0.989
3.6	0.021	0.993
3.8	0.015	0.995
4.0	0.011	0.997
4.5	0.005	0.999
5.0	0.000	1.000

Source: U.S. Department of Agriculture Soil Conservation Service (1972).

Figure 10: Ratios for dimensionless SCS synthetic unit hydrograph and mass curve.

Unit Hydrograph Derivation VIII

The SCS hydrograph is defined by three parameters: (1) Time of concentration, (2) Time to Peak, and (3) Peak Discharge.

Time of Concentration

As we previously discussed the time of concentration (t_c) for a watershed is the time for a water parcel to travel from the farthest point in the watershed to the outlet. The SCS recommends two methods for calculation of t_c:

▷ *The lag method*: defines the time lag as the time in hours from the center of the mass of the rainfall excess to the peak discharge, which can be computed as follows

$$t_L = \frac{L^{0.8}(S+1)^{0.7}}{1900Y^{0.5}}$$

where Y is the average slope of the watershed in %, L denotes the hydraulic length of the main channel in feet, and S represents the potential maximum retention. The SCS method suggests the following relationship for the concentration time

$$t_c = \frac{5}{3}\, t_L$$

▷ The upland or velocity method defines the concentration time as follows:

$$t_c = \frac{L}{3600V} \qquad [\text{hr}]$$

where L and V are hydraulic length and velocity in feet and ft/s, respectively. The velocity of the upland method can be obtained from Figure 8 based on the average slope of the watershed.

Unit Hydrograph Derivation IX

Clearly, if we discretized the the stream into k segments, the concentration time is the sum of travel time for different stream segments as we discussed before,

$$t_c = \frac{1}{3600} \Sigma_{i=1}^{k} \frac{L_i}{V_i} \qquad [\text{hr}]$$

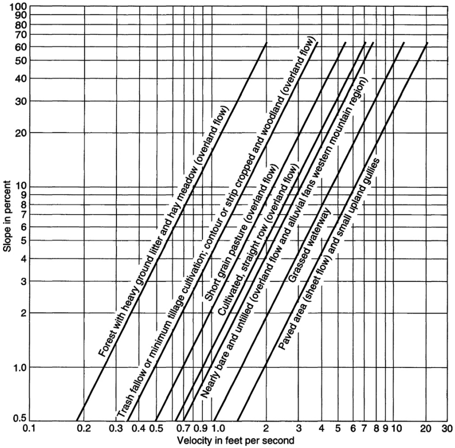

Figure 11: Velocities for upland method for estimation of t_c (USDA, 1986).

Unit Hydrograph Derivation X

Time to Peak

Time to peak (t_p) is the time from the beginning of the rainfall to the time of the peak discharge, which can be expressed as a function of the time lag t_L as follows:

$$t_p = \frac{t_R}{2} + t_L$$

where t_p is in hours, t_R is the duration of the rainfall excess in hours, and t_L is the lag time in hours. When t_R is not known, the SCS method suggests

$$t_R = 0.133 t_c$$

and because $t_L = 0.6 t_c$, one can obtain the following the following relationship for approximating the time to peak:

$$t_p = \frac{0.133 t_c}{2} + 0.6 t_c = 0.67 t_c$$

Peak Discharge

From continuity equation, we know that the area under the unit hydrograph is equal to the volume of the excess unit rainfall—example 1 [cm] or 1 [in] all over the watershed. Here, the unit of q_p in SCS method is the unit of discharge per depth of rainfall excess such as [m^3 s^{-1} cm^{-1}] or [cfs in^{-1}].

Unit Hydrograph Derivation XI

Peak Discharge (continued)

Therefore base on conservation of mass of the depth of the rainfall and the triangular shape of the proposed SCS unit hydrograph, we have

$$1 = \frac{1}{2}\frac{q_p}{A}(t_p + t_r).$$

which can be rearranged as

$$q_p = \left(\frac{2}{1 + t_r/t_p}\right)\frac{A}{t_p}$$

For the SI unit, in the above equation the area should be in [m^2] and time is in second. To convert the above formula to be used for watershed area in square kilometers and hourly time scale, we need to to multiply the above equation by $10^4/3600$. Moreover, the SCR method suggests $t_r = 1.67 t_p$ and thus

$$q_p = \frac{c\,A}{t_p}$$

where $c = (10000/3600) \times (2/2.67) = 2.08$. The c value in English unit is 483.4, where A should be in square miles.

Unit Hydrograph Derivation XII

Construction of SCS synthetic UH:

1: Compute the time of concentration (t_c) using the lag method or the upland-velocity method.

2: Compute the time to peak ($t_p = t_R/2 + 0.6t_c$) and then the peak discharge $q_p = cA/t_p$.

3: Compute the the base time of the SUH as $t_b = t_p + t_r$, which is $t_b = 2.67t_p$ for the triangular hydrograph and $t_b = 5t_p$ for the curvilinear one and the recession time $t_r = t_b - t_p$.

4: Compute the SHU ordinates and plot them. For the the triangular SUH, the ordinates can be directly computed while for the curvilinear one the dimensionless ratios (t/t_p, q/q_p) in Figure 8 shall be used.

Flow Routing I

Flow routing is the procedure of determining the hydrograph at a point from hydrographs at one or more points upstream. As water enters a channel, either from hillslopes or upstream channels, the flow velocity *shifts* the location of the peak flow and surface roughness and storages along the water path *attenuate* the magnitude of the peak. The flow routing is referred to a class of methods that calculates the redistribution (widening) and translation of upstream hydrograph as a function as the water flows downstream.

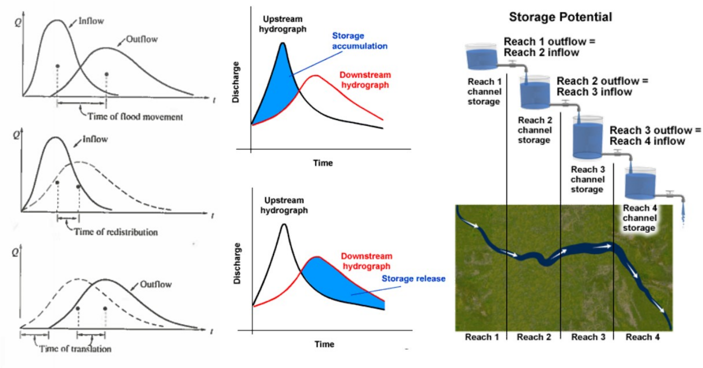

Figure 12: Left: Schematic of redistribution and translation from Chow et al. (1988). Center: The storage accumulation and release performed by routing. Right: Conceptualization of channels as storage reservoirs (Credit: Th COMET Program)

Flow Routing II

There are two main types of routing:

1. Lumped (Hydrologic) Routing: Flow is calculated at a particular point as a function of time alone.

2. Distributed (Hydraulic) Routing: Flow is calculated as a function of time and space.

For this course, we focus on the simpler lumped modeling.

Within lumped modeling, there are multiple methods depending on what system we are trying to model. We are going to focus on two common methods for flow routing in reservoirs (level-pool) and stream channels (Muskingum).

Derivation of routing methods begins with the continuity equation:

$$\frac{dS(t)}{dt} = I(t) - Q(t)$$

In routing, $I(t)$ is known inflow hydrograph, while $S(t)$ and $Q(t)$ are unknown. Just as in our discussion of linear reservoirs, we need a *storage-discharge* relationship, $S = f(Q)$.

There are two types of storage-discharge relationships: (1) invariable reservoirs and (2) variable reservoirs. The invariable are those in which there is a one-to-one relationship between the outflow and the reservoir storage. However, for variable reservoirs, the relationship between the outflow and storage capacity is not unique and follows a loop when the reservoir is filled or emptied. This nonlinear relationship mostly occurs in long and narrow reservoirs where the water velocity is significant and produces backwater effects in the reservoir.

Flow Routing III

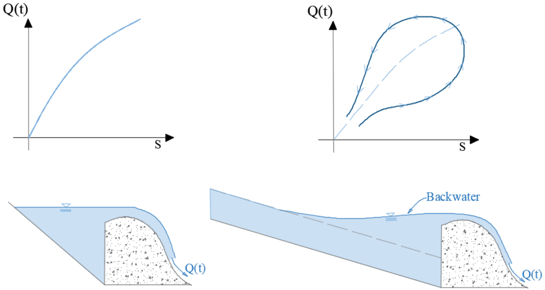

Figure 13: Invariable reservoir (left) that is wide and deep compared to its length. The flow velocity is low and the water surface is almost horizontal. A variable reservoir (right) is a long, narrow and shallow reservoir with high flow velocity such that the water surface profile is markedly curved due to the backwater effect.

Level Pool Routing

Level pool routing typically applies to man-made hydraulic structures like reservoirs, dams and stormwater ponds common in many watersheds. *The key assumption is a horizontal water surface and negligible velocity in the reservoir*. This allows an invariable relationship between storage (water surface height) and discharge, because any change in storage results in a change in the uniform water surface of the reservoir and in turn height of water above the flow control devices.

Beginning with the continuity equation $\frac{dS}{dt} = I(t) - Q(t)$ and thus $dS = I(t)dt - Q(t)dt$, one can discretize it as follows assuming that the changes in inflow and outflow are linear over over a small timestep, $\triangle t$:

Flow Routing IV

$$S_{j+1} - S_j = \frac{I_j + I_{j+1}}{2} \triangle t - \frac{Q_j + Q_{j+1}}{2} \triangle t$$

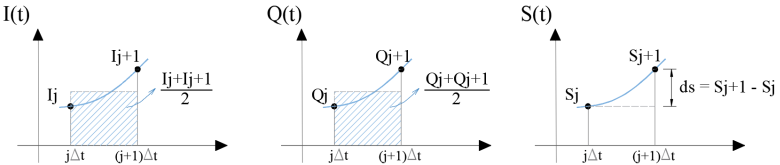

Figure 14: Discretization of the reservoir continuity equation over a small time interval.

$I(t)$ is known at all times while initial storage (S_j) and outflow (Q_j) are specified as initial conditions, so we can re-arrange the equation into knowns and unknowns as follows:

Level Pool Routing Equation

$$\left[\frac{2S_{j+1}}{\triangle t} + Q_{j+1}\right] = I_j + I_{j+1} + \left[\frac{2S_j}{\triangle t} - Q_j\right] \qquad (1)$$

Flow Routing V

Figure 15: Photos of a reservoir, dam and a pond with hydraulic structures as flow controls

To use this equation, *we need the elevation-storage and elevation-outflow relationships of the reservoir to create a storage-outflow function* $S = f(Q)$. The elevation-storage function $S = f(H)$ is simply constructed from the geometry of the reservoir or pond. The elevation-discharge $Q = f(H)$ is slightly more difficult to obtain, however, the outflow of storage facilities is often controlled by hydraulic structures such as sluice gates, weirs, pipes, and orifices which typically have well-defined equations based on water head (See Figure 15).

For example, for an uncontrolled ogee crest spillway the outflow as a function of total water head on the crest is:

$$Q = C_0 L H^{3/2}$$

L : is the effective length of the crest

C_0 : is the discharge coefficient

Q : is discharge in [cfs].

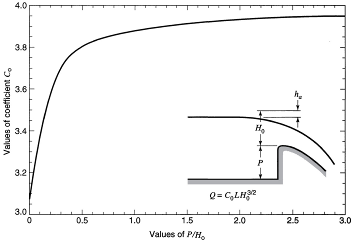

Figure 16: Discharge relationship for vertical-faced ogee crest (US bureau of reclamation 1987). The figure is for English system and h_a is the water surface upstream from weir drawdown.

Flow Routing VI

Once the storage-outflow function is calculated, as schematically shown in Figure 17, one can calculate the RHS of equation 1 and look up the LHS from the aforementioned table.

Additionally, there are more accurate methods to calculate level-pool routing, such a Runge-Kutta methods, if the assumption of inflow and outflow varying linearly at small timesteps is not acceptable.

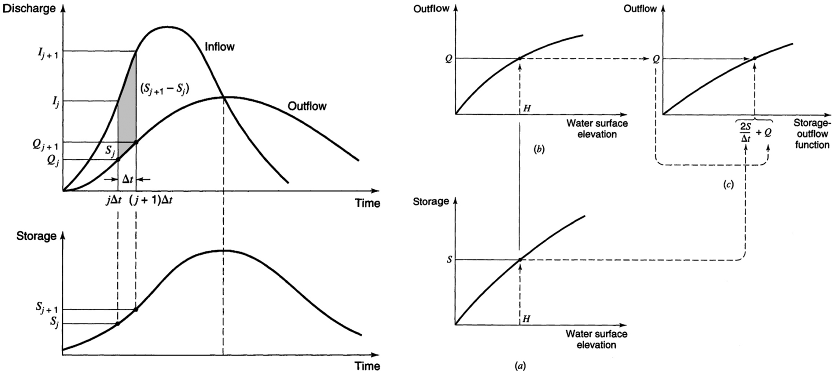

Figure 17: Change of storage during a routing period and development of the storage-outflow function for level pool routing on the basis of storage-elevation-outflow curves (Chow et al. 1988).

Flow Routing VII

Hydrologic River Routing (Muskingum Method)

In order to calculate the routing for streams in a watershed, we can no longer assume that the water surface is level, since there is not negligible velocity in a stream and there are backwater effects. Therefore, we need to define a different storage-outflow equation, $S = f(Q)$.

The Muskingum method assumes that storage in a channel is the sum of a prism and a wedge storage. The prism storage is a linear reservoir defined as $S = KQ$ where K is a constant representing approximately the travel time through the reach. The wedge storage is defined as $S = KX(I - Q)$, where X is a weighting factor ranging from 0 to 0.5 and thus,

$$S = KQ + KX(I - Q) = K(XI + (1 - X)Q)$$

This wedge storage is representing a flood wave as it propagates through a stream channel as shown in Figure 18.

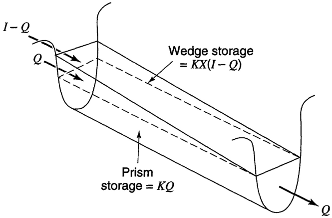

Figure 18: Schematics of wedge storage and floodwave propogation assumed in Muskingum method (Credit: The COMET Program)

Flow Routing VIII

We can discretize this storage equation as follows

$$S_j = K(XI_j + (1-X)Q_j) \qquad S_{j+1} = K(XI_{j+1} + (1-X)Q_{j+1})$$

and thus:

$$S_{j+1} - S_j = K\left\{[(XI_{j+1} + (1-X)Q_{j+1}] - [XI_j + (1-X)Q_j]\right\} \qquad (2)$$

Recall we previously defined the change in storage as:

$$S_{j+1} - S_j = \frac{I_j + I_{j+1}}{2}\triangle t - \frac{Q_j + Q_{j+1}}{2}\triangle t \qquad (3)$$

Combining equations 2 and 3 one can obtain the following flood routing equation:

Muskingum Equation for Stream Routing

$$Q_{j+1} = C_1 I_{j+1} + C_2 I_j + C_3 Q_j$$

$$C_1 = \frac{\triangle t - 2KX}{2K(1-X) + \triangle t} \qquad C_2 = \frac{\triangle t + 2KX}{2K(1-X) + \triangle t} \qquad C_3 = \frac{2K(1-X) - \triangle t}{2K(1-X) + \triangle t}$$

where $C_1 + C_2 + C_3 = 1$. Typically, the streams being analyzed are broken up into N subreaches when performing this calculation for stability purposes. The U.S. Army Corps of Engineers recommends the following condition to ensure solution stability:

$$\frac{1}{2(1/X)} \leq \frac{K}{N\triangle t} \leq \frac{1}{2X}$$