CEGE 4501 Hydrologic Design

Chapter 5: Infiltration

University of Minnesota
Driven to Discover[SM]

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Outline

Soil Properties I

"Soil" is a porous medium conisting of solid grains and **interconnected voids or pores** that can be filled with air or water.

The ability for soil to **retain and transport water** depends on both the distributions of the particle size and pore openings and the chemical composition of the soil particles themselves. A **"particle size distribution" (PSD)** is the most common tool used to classify soils according to standard criteria.

For larger grains (D $>$ 0.05 mm), **a series of successively smaller screens** are used to sieve out the mass percentage above each screen size. For smaller or fine grains (D $\leq$ 0.05 mm), the **settlement rate in a liquid** is used to determine the PSD.

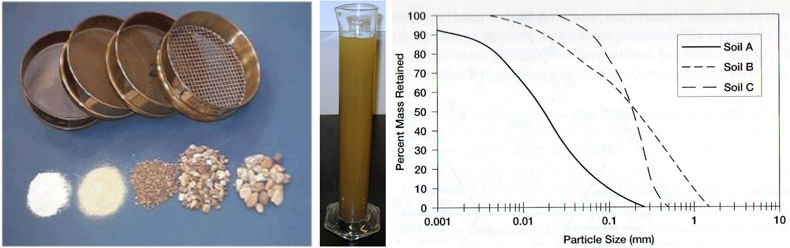


Figure 1: Left: Typical sieves used in creating a PSD (Credit: CivilBlog.org). Center: A typical hydrometer test for fine grains (Credit: Hoskin Scientifique). Right: A sample PSD for three different soils.

Soil Properties II

The primary particle size classification used by hydrologists is from the **United States Department of Agriculture (USDA)** (shown below). The range of grain sizes in a soil determines the **"soil texture"**, which is also defined by the USDA soil texture triangle below.

For example, **"Loam"** texture is defined as 40% sand, 40% silt, and 20% clay. Important to note that, fine grain soils (silt and clay) typically have **more total pore space** due to their larger surface area and mineral composition. However, as we will find out, this does **not** necessarily result in greater water transport through finer soils.

Class	Diameter (mm)
Gravel	>2
Sand	0.05–2
very coarse	1–2
coarse	0.5–1
medium	0.25–0.5
fine	0.10–0.25
very fine	0.05–0.10
Silt	0.002–0.05
Clay	<0.002

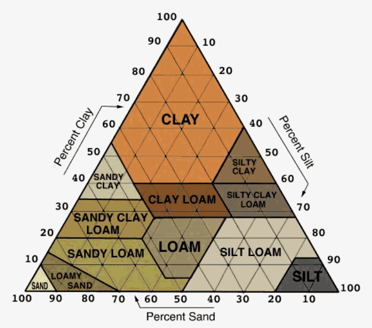


Figure 2: USDA soil particle (left) and soil texture (right) classifications.

Soil Properties III

From a hydrologic point of view, we are intersted in the **water holding capacity and transportation in soil**. Therefore, we need to define some basic mass and volume relationships to specify the amount of air, water and soil particles in a control volume.

Bulk Volume (V): The total control volume of the soil matrix, $V = V_v + V_s$, where V_v is the volume of voids and V_s is the volume of solids.

Volume of Voids (V_v): The total pore space of the soil matrix, $V_v = V_a + V_w$, where V_a is the volume of air and V_w is the volume of water.

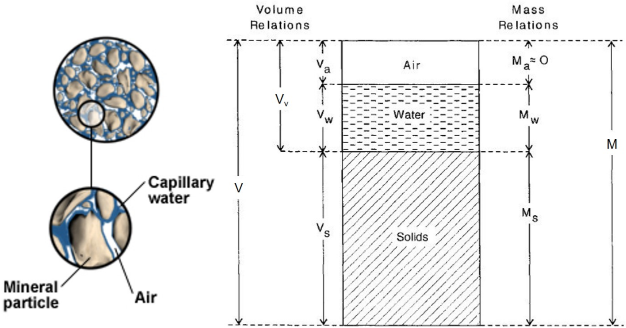


Figure 3: Classifying soil as a three phase system (air, water and solids) to create mass and volume relationships (Credit: The COMET Program and Hillel 1998)

Soil Properties IV

Porosity (η): Dimensionless measure of void space in soil matrix:

$$\eta = \frac{V_v}{V} = \frac{V - V_s}{V}$$

$$\eta = 1 - \frac{V_s}{V} = 1 - \frac{V_s/M_s}{V/M_s} = 1 - \frac{\rho_b}{\rho_s},$$

where M_s is the mass of the solids, ρ_b is the **"bulk density"**, which typically ranges from 1.0 to 1.6 g/cm^3, and ρ_s is the **"particle density"**, which typically ranges from 2.6 to 2.75 g/cm^3.

Note: There is a parameter called **"effective porosity" (θ_e or η_e)**, which excludes isolated pores and pore volume occupied by water adsorbed on clay minerals or other grains. Effective porosity is typically less than total porosity. We should emphesize that the total porosity is the total void space in the soil whether or not it contributes to fluid flow.

Volumetric Water Content (θ): Dimensionless measure of water in the soil matrix,

$$\theta = \frac{V_w}{V} \quad \text{where} \quad 0 \le \theta \le \eta$$

Relative Saturation (S): Volumetric water content normalized by porosity,

$$S = \frac{\theta}{\eta} \quad \text{where} \quad 0 \le S \le 1$$

Soil Properties V

SOIL TYPE	POROSITY
Sand	0.395 (0.056)
Loamy sand	0.410 (0.068)
Sandy Loam	0.435 (0.086)
Silt Loam	0.485 (0.059)
Loam	0.451 (0.078)
Sandy clay loam	0.420 (0.059)
Silty clay loam	0.477 (0.057)
Clay loam	0.476 (0.053)
Sandy clay	0.426 (0.057)
Silty clay	0.492 (0.064)
Clay	0.482 (0.050)

Soil class	Porosity η	Effective porosity θ_e
Sand	0.437 (0.374–0.500)	0.417 (0.354–0.480)
Loamy sand	0.437 (0.363–0.506)	0.401 (0.329–0.473)
Sandy loam	0.453 (0.351–0.555)	0.412 (0.283–0.541)
Loam	0.463 (0.375–0.551)	0.434 (0.334–0.534)
Silt loam	0.501 (0.420–0.582)	0.486 (0.394–0.578)
Sandy clay loam	0.398 (0.332–0.464)	0.330 (0.235–0.425)
Clay loam	0.464 (0.409–0.519)	0.309 (0.279–0.501)
Silty clay loam	0.471 (0.418–0.524)	0.432 (0.347–0.517)
Sandy clay	0.430 (0.370–0.490)	0.321 (0.207–0.435)
Silty clay	0.479 (0.425–0.533)	0.423 (0.334–0.512)
Clay	0.475 (0.427–0.523)	0.385 (0.269–0.501)

Figure 4: Left: Mean (standard deviation) of porosity as a function of soil type from Clapp & Hornberger (1978). Right: Mean (standard deviation) of porosity and effective porosity as a function of soil type from Rawls et al. (1983)

Vertical Distribution of Subsurface Water I

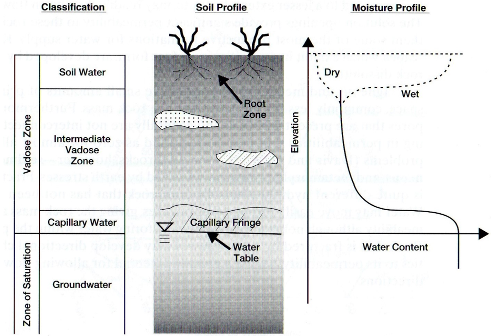


Figure 5: Classification of vertical soil structure based on the moisture content.

In hydrology, it is useful to conceptualize different zones in the subsurface based on their typical moisture profile as shown in Figure 5.

The **"saturated zone"** refers to sections where **pores are completely filled with water** and the pressure is **above atmospheric pressure**. This zone is typically what we think of as groundwater and follows simpler saturated flow equations. The groundwater table is where water pressure is equal to atmospheric pressure.

Vertical Distribution of Subsurface Water II

The **"vadose or unsaturated zone"** is the **partially saturated soil** directly above the groundwater table. The pore pressure in this region is **less than atmospheric pressure**. This zone is governed by more complex flow equations than the saturated zones and can be split into three additional sub-zones:

- **Soil-Water (Root) Zone:** This top section is crucial to hydrology as it considers the interaction of vegetation, soil texture and atmospheric conditions. During a precipitation event, this layer controls infiltration rate by its initial water content. After the storm, excess water is drained by gravity from this layer until the **"field capacity"** is reached. The soil moisture in this region can be evaporated directly from the soil surface or through transpiration. Therefore, this layer experiences significant diurnal and seasonal variability in moisture content.

- **Intermediate Vadose Zone:** Transition zone between the soil-water zone and capillary fringe. The soil moisture content in this layer does not change significantly in time.

- **Capillary Zone or Fringe:** The soil directly above the groundwater table where the pores are completely saturated but water is below atmospheric pressure due to water being pulled up into the pores because of an adhesion force, called capillary forces, between water molecules and soil particles. The magnitude of the capillary force depends on the pore size distribution and thus soil texture. This pressure is much higher in fine grain soils, such as clay, than those soils with coarser particles such sand.

Material	Grain Size (mm)	Capillary Rise (cm)
Fine gravel	2–5	2.5
Very coarse sand	1–2	6.5
Coarse sand	0.5–1	13.5
Medium sand	0.2–0.5	24.6
Fine sand	0.1–0.2	42.8
Silt	0.05–0.1	105.5
Silt	0.02–0.05	200[1]

[1]Still rising after 72 days

Figure 6: Capillary rise in soils with similar porosity after 72 days (from Lohman, 1972).

Saturated Flow in Porous Media I

Hydraulic Head

Water flows down the prevailing energy gradient which we can define at a given point in a fluid flow field in terms of:

Potential Energy: mgz **Kinetic Energy**: $\frac{mv^2}{2}$ **Pressure Energy**: $pV = p\frac{m}{\rho_w}$

The total energy of the fluid flow is thus

$$E = mgz + \frac{mv^2}{2} + p\frac{m}{\rho_w} \qquad [Joules]$$

but it is often convenient to express the components in terms of **"hydraulic head"** which is total energy per unit weight of a water parcel as follows:

$$\frac{E}{mg} = h = z + \frac{v^2}{2g} + \frac{p}{\rho_w g} \qquad [m].$$

Here, the pressure head (p) is the gauge pressure, which means that we should not account for atmospheric pressure.

In subsurface flows, the velocity is typically very slow and we can assume $v^2/2g \to 0$ so,

$$h = z + \frac{p}{\rho_w g} \qquad [m]$$

Therefore, for saturated flows, the hydraulic head driving the flow is dependent on **gravitational forces** and **hydrostatic pressure forces**. With this in mind we will now cover the saturated flow equation known as Darcy's Law.

Saturated Flow in Porous Media II

Darcy's Law for Saturated Flow

In 1856, engineer Henry Darcy ran experiments similar to Figure 7 to test sand filters for purifying the drinking water of Dijon, France.

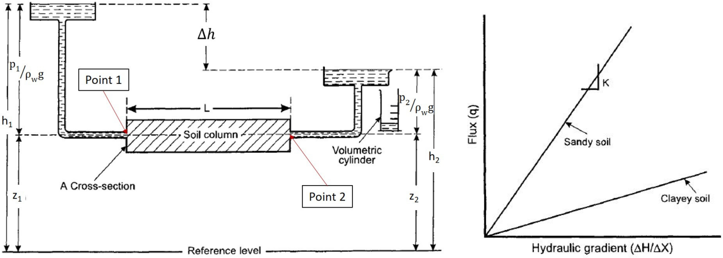


Figure 7: Left: Experimental setup for measuring flow through a horizontal column (Hillel, 1998).

Darcy found that the **steady state flux** (q) from the experimental setup was **proportional to the hydraulic gradient** across the filter length ($\frac{\triangle h}{L}$) as follows:

Darcy's Law

$$q = \frac{Q_{out}}{A} = K_{sat} \cdot \frac{\triangle h}{L} \qquad [\mathrm{m^3 s^{-1} m^{-2}}]$$

Saturated Flow in Porous Media III

where K_{sat} [m s^{-1}] is the constant of proportionality known as the (saturated)**"hydraulic conductivity"**.

As an example, in Figure 7, we could define points at the entrance and exist of the soil column and Darcy's Law would read:

$$q = K_{sat} \cdot \frac{h_1 - h_2}{L} = K_{sat} \cdot \frac{(z_1 + \frac{p_1}{\rho_w g}) - (z_2 + \frac{p_2}{\rho_w g})}{L}$$

however, $z_1 = z_2$ so,

$$q = K_{sat} \cdot \frac{\frac{p_1}{\rho_w g} - \frac{p_2}{\rho_w g}}{L}$$

As is shown in Figure 7, the hydraulic conductivity **linearly transforms** the hydraulic gradient to a flux and is determined by soil. Some typical values are shown on the next slide, **varying by orders of magnitude**. As a general rule, soils with higher clay content have much lower K_{sat} values given there is a greater piezometric head loss for a given flux due to the more tortuous flow path of the smaller pores.

Soil Type	K_{sat} [cm h^{-1}]	Soil Type	K_{sat} [cm h^{-1}]
Sand	63.36	Clay loam	0.88
Loamy sand	56.16	Silty clay loam	0.61
Sandy loam	12.49	Sandy clay	0.78
Loam	2.50	Silty clay	0.37
Silt loam	2.59	Clay	0.46
Sandy clay loam	2.27		

Table 1: The hydraulic conductivity for different USDA soil textures (Clapp & Hornberger, 1978))

Unsaturated Flow in Porous Media I

Darcy's Law for Unsaturated Flow

Darcy's Law was originally created for saturated flow below the water table, however, many important processes to hydrology occur in the vadose zone.

In this unsaturated zone, we must deal with pressure that is below atmospheric due to **suction or capillary action**. Recall from fluid mechanics, the attraction of water to a surface, adhesion, and to itself, cohesion, causes water to be pulled up a narrow capillary tube until that suction force is balanced by the gravitational pull of the column of water. The height of capillary rise is represented by:

$$h_c = \frac{2\sigma cos(\theta)}{r \cdot \gamma w}$$

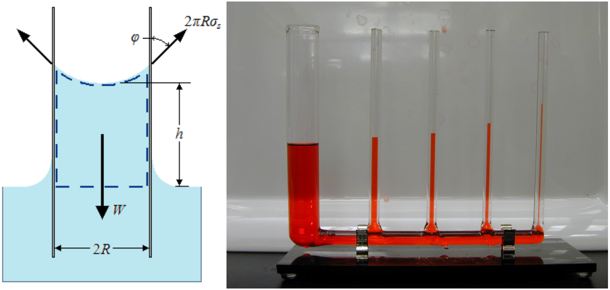


Figure 8: Conceptual and actual capillary tubes (Credit: Hillel (1998)).

Unsaturated Flow in Porous Media II

Now we can think of the tortuous pores in soils as a series of capillary tubes (even though that's not quite true), and each will have a suction pressure or capillary pressure. The smaller the pores, the higher the capillary rise, therefore **clays will have a higher rise than sands**. Let us represent the **"suction head"** as a pressure head:

$$\psi = \frac{p}{\rho_w g} < 0 \quad [L]$$

Material	Grain Size (mm)	Capillary rise (cm)
Fine gravel	2-5	2.5
Very coarse sand	1-2	6.5
Coarse sand	0.5-1	13.5
Medium sand	0.2-0.5	24.6
Fine sand	0.1-0.2	42.8
Silt	0.05-0.1	105.5
Silt	0.02-0.05	200

Table 2: Capillary rise in soils with similar porosity after 72 days (from Lohman, 1972)

From here we can modify what we consider the hydraulic or **"piezometric head"** as follows:

$$h = \psi + z$$

Now let us define Darcy's Law for unsaturated flow with this new definition of piezometric head. For simplicity, we are going to define it only in z-direction since that will be our primary focus for studying infiltration.

$$q_z = -K_z \cdot \frac{\partial h}{\partial z} = -K_z \cdot (\frac{\partial \psi}{\partial z} + \frac{\partial z}{\partial z}) = -K_z \cdot (\frac{\partial \psi}{\partial z} + 1)$$

Unsaturated Flow in Porous Media III

Now we know from experience that hydraulic conductivity and suction head are function of soil moisture content θ. For a drier soil, there is more surface area on particles available for water to adhere to, creating a very large suction head, whereas there are fewer saturated pores to transmit water so the hydraulic conductivity decreases. As the soil gets wetter, there is less surface area attracting water while there are more wetted porous pathways to transmit water, therefore suction head decreases while hydraulic conductivity increases. Knowing this we can use the chain rule to write Darcy's Law as follows:

$$q_z = -K_z(\theta) \cdot \left(\frac{\partial \psi(\theta)}{\partial \theta} \frac{\partial \theta}{\partial z} + 1 \right) = -\left(K_z(\theta) \cdot \frac{\partial \psi(\theta)}{\partial \theta} \frac{\partial \theta}{\partial z} + K_z(\theta) \right)$$

Now for simplicity we can define the first term on the left hand side as a function of θ called the the **"soil water diffusivity"** $D(\theta) = K_z(\theta) \frac{\partial \psi(\theta)}{\partial \theta}$ to get our final form of Darcy's Law for unsaturated flow:

Darcy's Law for Unsaturated Steady Flow

$$q_z = -\left(D(\theta) \cdot \frac{\partial \theta}{\partial z} + K_z(\theta) \right) \qquad [\mathrm{m^3\, s^{-1}\, m^{-2}}]$$

Unsaturated Flow in Porous Media IV

The moisture flux in the unsaturated zone is driven by two factors: 1) the metric forces (from wetter to drier soil) and 2) gravity (always downward). To get a better understanding, the figure below shows the signs of these terms in the equation for two ideal scenarios. Note that we here define z as positive upward.

1) Soil is wet near the surface and drier beneath as would happen during a rainfall event.

$$-D(\theta)\cdot\frac{\partial\theta}{\partial z} < 0 \downarrow \ \textit{since}\ \theta_2-\theta_1>0 \qquad \textit{and} \qquad -K(\theta)<0\downarrow$$

2) Soil is dry near the surface and wetter beneath as would happen during a hot and dry day.

$$-D(\theta)\cdot\frac{\partial\theta}{\partial z} > 0 \uparrow \ \textit{since}\ \theta_2-\theta_1<0 \qquad \textit{and} \qquad -K(\theta)<0\downarrow$$

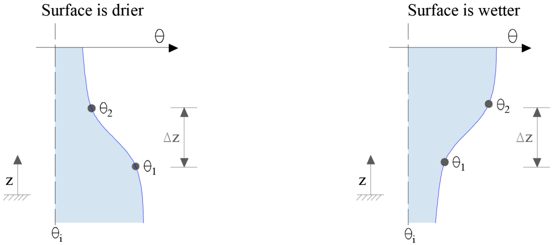


Figure 9: Conceptual sketch of dominant forces in unsaturated flow.

Unsaturated Flow in Porous Media V

Soil Water Characteristic Curve

In order to use the unsaturated Darcy Law, we need curves for both $\psi(\theta)$ and $K_z(\theta)$, which are most accurately determined from lab tests. The**"soil water characteristic curve (SWC)"** or retention curve is the relationship between θ and $\psi(\theta)$ which can be determined using tension or pressure plate experiments shown on the next slide. Figure 12 shows that **the lower the water content the higher the suction head**, as previously mentioned, and that larger grains typically hold less water overall and through their profiles.

The water content vs. unsaturated hydraulic conductivity is determined similar to K_{sat} by measuring outflow for as set of suction head differences. The SWC curve is then used to assign a water content to each $K(\theta)$ value. **Note: Once the suction head is zero (or soil is saturated), $K(\theta)$ goes to K_{sat} .**

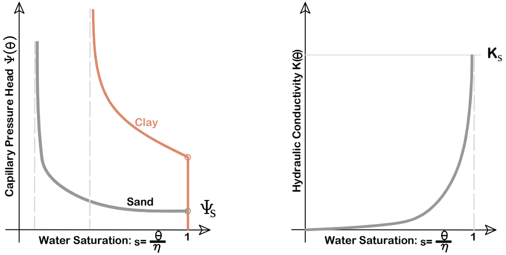


Figure 10: Schematic representation of changes in capillary pressure and hydraulic conductivity as a function relative saturation $s = \theta/\eta$.

Unsaturated Flow in Porous Media VI

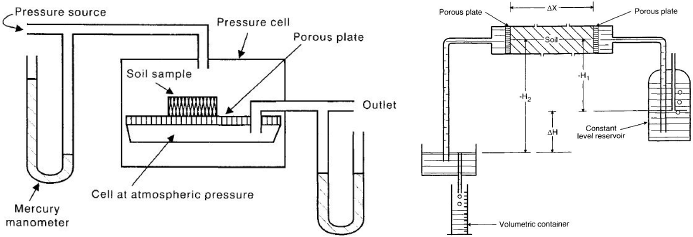


Figure 11: Typical pressure plate setup (left) for determining SWC and flow test setup for determining unsaturated hydraulic conductivity (right; Credit: Hillel, 1998).

There have been multiple attempts to parameterize the SWC and K_{sat} curves based on soil texture or more easily measured parameters. One of the simpler methods used widely in hydrology is the "Brooks-Corey model" (1966):

$$\psi(\theta) = \psi_{\text{sat}} \left(\frac{\theta - \theta_r}{\eta - \theta_r} \right)^{-b} \qquad \text{and} \qquad K(\theta) = K_{\text{sat}} \left(\frac{\theta - \theta_r}{\eta - \theta_r} \right)^{c}$$

ψ_{sat}: Saturated capillary pressure [L]

K_{sat}: Hydraulic conductivity [L T^{-1}]

η: Porosity

θ_r: Irreducible soil moisture content (if not zero)

b: Brooks-Corey's parameter and $c = 2b + 3$.

Unsaturated Flow in Porous Media VII

It should be noted that there are other more complex models, like the Mualem-van Genuchten model, but the Brooks-Corey model is widely used in hydrology with some typical parameter values shown in the following table.

Soil Type	ψ_{sat} [cm]	b (Brooks-Corey)
Sand	-12.1 (14.3)	4.05 (1.78)
Loamy sand	-9.00 (12.4)	4.38 (1.47)
Sandy loam	-21.8 (31.0)	4.90 (1.75)
Loam	-47.8 (51.2)	5.39 (1.87)
Silt loam	-78.6 (51.2)	5.30 (1.96)
Sandy clay loam	-29.9 (37.8)	7.12 (2.43)
Clay loam	-63.0 (51.0)	8.52 (3.44)
Silty clay loam	-35.6 (37.8)	7.75 (2.77)
Sandy clay	-15.3 (17.3)	10.4 (1.64)
Silty clay	-49.0 (62.1)	10.4 (4.45)
Clay	-40.5 (39.7)	11.4 (3.70)

Table 3: Saturated matric head ψ_{sat} and Brooks-Corey (b) (from Clapp & Hornberger, 1978) parameter. The values in parentheses represent the uncertainty in terms of one standard deviation.

Typical Soil Water Conditions

Field Capacity θ_{fc} : This is the maximum water content that can be retained by a soil against gravitational forces. A vague but practical explanation of this quantity is the moisture content of a soil 2 to 3 days after rainfall or irrigation. A more strict definition is that field capacity is the water content corresponding to $\psi_{fc} = -340\, cm$.

Unsaturated Flow in Porous Media VIII

Wilting Point θ_{wp}: This is the defined water content at which most plants will begin to wilt since they cannot access the water they need from the soil. It is formally defined as the water content at $\psi_{wp} = -1500\,cm$.

Available water content θ_a: The difference between field capacity and the wilting point which defines how much water plants have access to:

$$\theta_a = \theta_{fc} - \theta_{wp}$$

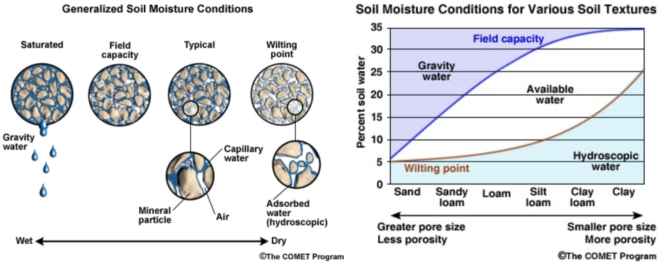


Figure 12: Left: Illustration of various wetness conditions. Right: Graphical representation of wetness conditions for different soil textures (Credit: The COMET Program)

Unsaturated Flow in Porous Media IX

Richard's Equation

Thus far we have been discussing steady state flow equations, however, in order to define the governing equation of unsaturated flow, we need to incorporate time through the **conservation of mass**. We can derive this by looking at a cube control volume and assume that the only flux is in the z-direction (which is not a terrible assumption):

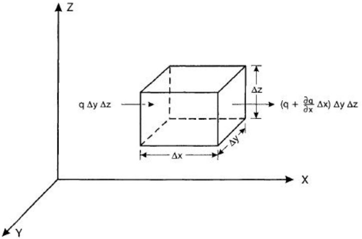


Figure 13: Control volume element used to derive continuity equation for unsaturated flow.

Accumulation of (mass/time) = (Input mass-Output mass)/time $\pm$ Source/Sink

Accumulation of (mass/time) $= \frac{\partial \rho_w \theta}{\partial t} \cdot dx\, dy\, dz$

Input (mass/time) $= \rho_w \cdot q_z \cdot dx\, dy$

Output (mass/time) $= \rho_w \left(q_z + \frac{\partial q_z}{\partial z} dz \right) \cdot dx\, dy$

Source/Sink $= \rho_w \cdot S(z, t) \cdot dx\, dy\, dz$

Unsaturated Flow in Porous Media X

where all terms have units of [kg s^{-1}]. $S(z,t)$ is a term that can be used to represent plant water uptake as a sink ($S(z,t) < 0$) or other sources ($S(z,t) > 0$). Combining these equations leaves the following continuity equation:

$$\frac{\partial \theta}{\partial t} = -\frac{\partial q_z}{\partial z} + S(z,t)$$

Note: This conservation equation has unit of [T^{-1}] since it is (water volume)/(total volume)×(time)$^{-1}$.

Finally, we are able to substitute the unsaturated Darcy's Law (momentum equation) for the q_z term in the continuity equation to obtain the one-dimensional Richard's Equation:

One-Dimensional Richard's Equation (1931)

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta)\cdot\frac{\partial \theta}{\partial z} + K_z(\theta)\right) + S(z,t) \qquad \left[T^{-1}\right]$$

Note: that $\theta(z,t)$ and thus $K(\theta)$ and $D(\theta)$ are functions of z and thus we can not take $D(\theta)$ out from the derivative operator. This makes the Richard's equation a non-linear diffusion equation. Since $D(\theta)$ is extremely nonlinear with respect to θ, the solutions of the Richards equation through numerical methods are often challenging and subject to numerical instabilities.

Infiltration Models I

We have now derived a detailed equation for unsteady, unsaturated flow in porous medium. However, in hydrology, we are typically interested with subsurface flow in terms of how much precipitation infiltrates at the surface into the vadose zone. Given the difficulties in numerically solving the Richard's equation over a large watershed, it is more practical to make simplified assumptions and for calculating the surface downward water fluxes:

Conceptual Understanding of Infiltration

From earlier discussion, the infiltration flux at the earth's surface $z = 0$ can be written as follows:

$$f(t) = q_z\Big|_{z=0} = -D(\theta_{0,t}) \cdot \frac{\partial \theta_{0,t}}{\partial z} \pm K_z(\theta_{0,t}) \qquad [\mathrm{m^3\,s^{-1}\,m^{-2} \text{ or } m\,s^{-1}}]$$

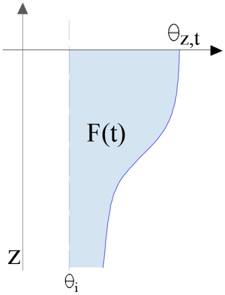
where $\theta_{0,t}$ is shorthand for $\theta(z,t)$ at $z = 0$ and time t. To reiterate, $K_z(\theta_{0,t})$ has a negative sign when z is positive upward and has a positive sign when z is positive downward.

We can then simply define the cumulative infiltration by integrating the infiltration rate over time or over depth:

$$F(t) = \int_0^t f(t)dt \qquad or \qquad F(t) = \int_0^\infty (\theta_{z,t} - \theta_i)dz$$

where θ_i is the initial soil moisture content of the soil.

Based on this understanding, let's define three infiltration scenarios that may occur during a precipitation event of rate p [mm hr^{-1}]. For brevity, we will omit the subscripts specifying that we are at $z = 0$ and $t = 0$, but keep it in mind.



Infiltration Models II

1) Ponding after a certain time ($p < f(t=0)$):

$$f(t) = \begin{cases} p & t \leq t_p \\ -D(\theta_{sat}) \cdot \frac{\partial \theta_{sat}}{\partial z} + K_{z=0}(\theta_{sat}) & t > t_P \end{cases}$$

where t_p is the **"time to ponding"** at after which water accumulates on the ground and the soil at z=0 is assumed to be saturated. Note, that after the ponding time, the equations assume the values of θ and $K_z(\theta)$ correspond to the saturated values. As time passes the diffusive element, $\partial \theta_{sat} / \partial z$ diminishes leaving only the gravitational component $K_{z=0}(\theta_{sat})$. In other words, $f(t) \rightarrow K_{sat}$.

2) Immediate Ponding ($p \gg f(t=0)$):

$$f(t) = -D(\theta_{sat}) \cdot \frac{\partial \theta_{sat}}{\partial z} + K_z(\theta_{sat}) \qquad t \geq 0$$

3) No Ponding ($p \ll f(t=0)$):

$$f(t) = p \qquad t \geq 0$$

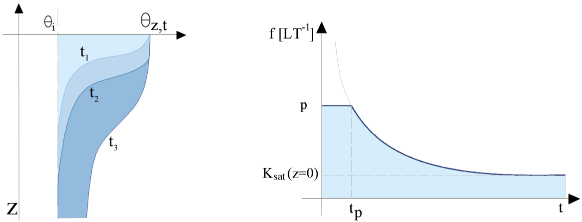


Figure 14: Typical water content profile (left) and infiltration rate (right) evolution.

Infiltration Models III

Philip's Model (1960)

Philip (1960) provided an approximate solution to Richard's equation using a series expansion to eventually approximate infiltration rate as:

$$f(t) = \frac{S_1}{2} t^{-1/2} + K_{sat}$$

where S_1 is called the sorptivity coefficient:

$$S_1 = (n - \theta_i) K_{sat} |\psi_f| \qquad [in/\sqrt{hr}]$$

where $|\psi_f|$ is the suction head at the edge of the wetting from defined by:

$$|\psi_f| = \frac{2b+3}{b+3} |\psi_s|$$

and $|\psi_s|$ is the saturated capillary head, b denotes the Brooks-Corey's constant. Integrating Philip's equation yields the cumulative infiltration function as follows:

$$F(t) = \int_{t_0}^{t} f(t) dt = S_1 (t - t_0)^{-1/2} + K_{sat}(t - t_0).$$

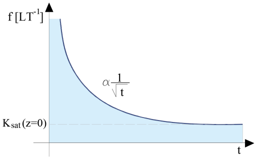


Figure 15: Phillip's model assumes that infiltration rate decays as square root of t. we need to note that the Philip's model is undefined at $t = 0$.

Infiltration Models IV

Green-Ampt Model (1911)

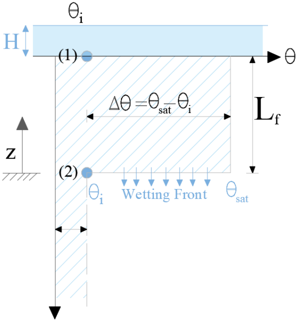


Figure 16: Conceptual diagram of the wetting frong and variables used in the Green-Ampt model.

For Green-Ampt we assume a sharp wetting front (Figure 17) and we begin with Darcy's Law for saturated flow:

$$q_z = -K_{sat} \cdot \frac{dh}{dz} = -K_{sat} \cdot \frac{h_2 - h_1}{L_f} = -K_{sat} \cdot \frac{-|\psi_f| - (L_f + H)}{L_f}$$

$$q_z = f(t) = K_{sat} \cdot \frac{|\psi_f| + L_f + H}{L_f}$$

Infiltration Models V

where L_f is the "length of the wetting front" and H is the "ponding depth at the surface". Now using this equation we can make a few assumptions in order to calculate the time to ponding if the rainfall rate p is less than infiltration capacity at $t = 0$. Let's substitute p for q_z and assume that H is negligible.

$$p = K_{sat}\left(1 + \frac{|\psi_f|}{L_f}\right)$$

Then we can assume that the cumulative depth of infiltrated water is defined with the sharp wetting front as:

$$F = L_f\left(\theta_s - \theta_i\right) = L_f \triangle\theta$$

and that the cumulative infiltration at the time to ponding can be represented by $F_p = p \cdot t_p$:

$$p = K_{sat}\left(1 + \frac{|\psi_f|\triangle\theta}{F_p}\right) = K_{sat}\left(1 + \frac{|\psi_f|\triangle\theta}{p \cdot t_p}\right)$$

$$t_p = \frac{K_{sat}|\psi_f|\triangle\theta}{p(p - K_{sat})}$$

Now let's create an equation for cumulative infiltration when $t > t_p$. There are two separate definitions that can be created and set equal if we assume H is negligible:

$$f(t) = \triangle\theta\frac{dL_f}{dt} \qquad f(t) = K_{sat}\left(1 + \frac{|\psi_f|}{L_f}\right)$$

$$\triangle\theta\frac{dL_f}{dt} = K_{sat}\left(1 + \frac{|\psi_f|}{L_f}\right)$$

$$\frac{K_{sat}}{\triangle\theta} \cdot dt = \frac{L_f}{L_f + |\psi_f|} \cdot dL_f$$

Infiltration Models VI

Now we also know that $F(t) = \triangle\theta L_f(t)$

$$\frac{K_{sat}}{\triangle\theta} \cdot dt = \frac{F(t)/\triangle\theta}{F(t)/\triangle\theta + |\psi_f|} \cdot \frac{dF(t)}{\triangle\theta}$$

$$K_{sat} \cdot dt = \frac{F(t)}{F(t) + \triangle\theta|\psi_f|} \cdot dF(t)$$

Then upon integration we can obtain:

$$F(t) = Kt + |\psi_f| \cdot \triangle\theta \cdot ln\left(1 + \frac{F(t)}{|\psi_f|\triangle\theta}\right)$$

The reason we want cumulative infiltration as a function of time is because we can use it to calculate runoff:

$$Runoff = R(t) = pt - F(t)$$

Infiltration Models VII

Summary of the Green-Ampt Model

Ponding time: $t_p = \frac{K_{sat}\Delta\theta|\psi_f|}{p(p-K_{sat})}$ [T]

Cumulative infiltration before ponding time:

$F(t) = pt \quad t < t_p \quad \text{and} \quad F(t_p) = pt_p$

Cumulative infiltration at ponding time:

$$F(t_p) = K_{sat}\,t_p + \Delta\theta\,|\psi_f|\ln\left(1 + \frac{F(t_p)}{|\psi_f|\Delta\theta}\right) \quad (1)$$

Cumulative infiltration at $t_p + \Delta t$:

$$F(t_p + \Delta t) = K_{sat}\,(t_p + \Delta t) + \Delta\theta\,|\psi_f|\ln\left(1 + \frac{F(t_p+\Delta t)}{|\psi_f|\Delta\theta}\right) \quad (2)$$

(2)-(1):

$$F(t_p + \Delta t) = F(t_p) + K_{sat}\,(\Delta t) + \Delta\theta\,|\psi_f|\ln\left(\frac{F(t_p+\Delta t)+\Delta\theta|\psi_f|}{F(t_p)+|\psi_f|\Delta\theta}\right)$$